Unit 3 – Test

1. If the terminal side is on \((-8, 15)\), then the hypotenuse of this triangle would be 17 (by the Pythagorean Theorem, \(c = \sqrt{(-8)^2 + 15^2}\)). Therefore, \(\sin x = \frac{15}{17}\), \(\cos x = -\frac{8}{17}\), and \(\tan x = -\frac{15}{8}\).

2. If \(\sin a = \frac{\sqrt{3}}{2}\) and \(\tan a < 0\), then \(a\) is in Quadrant II. Therefore, \(\sec a\) is negative. To find the third side, we need to do the Pythagorean Theorem.

\[
\left(\sqrt{3}\right)^2 + b^2 = 3^2 \\
3 + b^2 = 9 \\
b^2 = 6 \\
b = \sqrt{6}
\]

So see \(a = -\frac{\sqrt{2}}{2}\).

3. Factor top, cancel like terms, and use the Pythagorean Theorem Identity. Note that this simplification doesn’t hold true for values of \(x\) that are \(\frac{\pi}{4} + \frac{\pi n}{2}\), where \(n\) is a positive integer, since the original expression is undefined for these values of \(x\).

\[
\frac{\cos^4 x - \sin^4 x}{\cos^2 x - \sin^2 x} \\
\frac{\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)}{\cos^2 x - \sin^2 x} \\
\cos^2 x + \sin^2 x \\
1 \\
1 \\
1
\]

4. Change secant and cosecant into terms of sine and cosine, then find a common denominator.

\[
\frac{1 + \sin x}{\cos x \sin x} = \sec x (\csc x + 1) \\
= \frac{1}{\cos x} \left( \frac{1}{\sin x} + 1 \right) \\
= \frac{1}{\cos x} \left( \frac{1 + \sin x}{\sin x} \right) \\
= \frac{1 + \sin x}{\cos x \sin x}
\]

5.

\[
\sec \left(x + \frac{\pi}{2}\right) + 2 = 0 \\
\sec \left(x + \frac{\pi}{2}\right) = -2 \\
\cos \left(x + \frac{\pi}{2}\right) = -\frac{1}{2} \\
x + \frac{\pi}{2} = \frac{2\pi}{3}, \frac{4\pi}{3} \\
x = \frac{2\pi}{3} - \frac{\pi}{2}, \frac{4\pi}{3} - \frac{\pi}{2} \\
x = \frac{5\pi}{6}, \frac{\pi}{6}
\]
6.

\[ 8 \sin\left(\frac{x}{2}\right) - 8 = 0 \]
\[ 8 \sin\frac{x}{2} = 8 \]
\[ \sin\frac{x}{2} = 1 \]
\[ \frac{x}{2} = \pi \]
\[ x = 2\pi \]

7.

\[ 2 \sin^2 x + \sin 2x = 0 \]
\[ 2 \sin^2 x + 2 \sin x \cos x = 0 \]
\[ 2 \sin x (\sin x + \cos x) = 0 \]
So, \( 2 \sin x = 0 \) or \( \sin x + \cos x = 0 \)
\[ 2 \sin x = 0 \]
\[ \sin x = 0 \]
\[ x = 0, \pi \]
\[ \sin x + \cos x = 0 \]
\[ \sin x = -\cos x \]
\[ x = \frac{3\pi}{4}, \frac{7\pi}{4} \]

8.

\[ 3 \tan^2 2x = 1 \]
\[ \tan^2 2x = \frac{1}{3} \]
\[ \tan 2x = \pm \frac{\sqrt{3}}{3} \]
\[ 2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6} \]
\[ x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12} \]

9.

\[ 1 - \sin x = \sqrt{3} \sin x \]
\[ 1 = \sin x + \sqrt{3} \sin x \]
\[ 1 = \sin x \left(1 + \sqrt{3}\right) \]
\[ \frac{1}{1 + \sqrt{3}} = \sin x \]

\[ \sin^{-1}\left(\frac{1}{1 + \sqrt{3}}\right) = x \] or \( x = 0.3747 \) radians and \( x = 2.7669 \) radians

10. Because this is \( \cos 3x \), you will need to divide by 3 at the very end to get the final answer. This is why we
went beyond the limit of $2\pi$ when finding $3x$.

$$2 \cos 3x - 1 = 0$$
$$2 \cos 3x = 1$$
$$\cos 3x = \frac{1}{2}$$

$$3x = \cos^{-1} \left( \frac{1}{2} \right) = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}$$

$$x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$$

11. Rewrite the equation in terms of $\tan$ by using the Pythagorean identity, $1 + \tan^2\theta = \sec^2\theta$.

$$2 \sec^2 x - \tan^4 x = 3$$
$$2(1 + \tan^2 x) - \tan^4 x = 3$$
$$2 + 2 \tan^2 x - \tan^4 x = 3$$
$$\tan^4 x - 2 \tan^2 x + 1 = 0$$
$$(\tan^2 x - 1)(\tan^2 x - 1) = 0$$

Because these factors are the same, we only need to solve one for $x$.

$$\tan^2 x - 1 = 0$$
$$\tan^2 x = 1$$
$$\tan x = \pm 1$$

$$x = \frac{\pi}{4} + \pi k \text{ and } \frac{3\pi}{4} + \pi k$$

Where $k$ is any integer.

12. Use the half angle formula with $315^\circ$.

$$\cos 157.5^\circ = \cos \frac{315^\circ}{2}$$
$$= -\sqrt{\frac{1 + \cos 315^\circ}{2}}$$
$$= -\sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}}$$
$$= -\sqrt{\frac{2 + \sqrt{2}}{4}}$$
$$= -\frac{\sqrt{2 + \sqrt{2}}}{2}$$

13. Use the sine sum formula.

$$\sin \frac{13\pi}{12} = \sin \left( \frac{10\pi}{12} + \frac{3\pi}{12} \right)$$
$$= \sin \left( \frac{5\pi}{6} + \frac{\pi}{4} \right)$$
$$= \sin \frac{5\pi}{6} \cos \frac{\pi}{4} + \cos \frac{5\pi}{6} \sin \frac{\pi}{4}$$
$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + (\frac{-\sqrt{3}}{2}) \cdot \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$
14. 
\[
4(\cos 5x + \cos 9x) = 4 \left[ 2 \cos \left( \frac{5x + 9x}{2} \right) \cos \left( \frac{5x - 9x}{2} \right) \right] \\
= 8 \cos 7x \cos (-2x) \\
= 8 \cos 7x \cos 2x
\]

15. 
\[
cos(x - y) \cos y - \sin(x - y) \sin y \\
\cos y(\cos x \cos y + \sin x \sin y) - \sin y (\sin x \cos y - \cos x \sin y) \\
\cos x \cos^2 y + \sin x \sin y \cos y - \sin y \sin^2 x \cos y + \cos y \sin^2 y \\
\cos x \cos^2 y + \cos x \sin^2 y \\
\cos x (\cos^2 y + \sin^2 y)
\]

16. Use the sine and cosine sum formulas.
\[
\sin \left( \frac{4\pi}{3} - x \right) + \cos \left( x + \frac{5\pi}{6} \right) \\
= \sin \left( \frac{4\pi}{3} \right) \cos x - \cos \left( \frac{4\pi}{3} \right) \sin x + \cos x \cos \left( \frac{5\pi}{6} \right) - \sin x \sin \left( \frac{5\pi}{6} \right) \\
- \frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x - \frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x \\
- \sqrt{3} \cos x
\]

17. Use the sine sum formula as well as the double angle formula.
\[
\sin 6x = \sin (4x + 2x) \\
\sin 4x \cos 2x + \cos 4x \sin 2x \\
= \sin (2x + 2x) \cos 2x + \cos (2x + 2x) \sin 2x \\
= \cos 2x (\sin 2x \cos 2x + \cos 2x \sin 2x) + \sin 2x (\cos 2x \cos 2x - \sin 2x \sin 2x) \\
= 2 \sin 2x \cos^2 2x + \sin 2x \cos^2 2x - \sin^2 2x \\
= 3 \sin 2x \cos^2 2x - \sin^2 2x \\
= 2 \sin 2x (3 \cos^2 2x - \sin^2 2x) \\
= 2 \sin x \cos x \left[ (\cos^2 x - \sin^2 x)^2 - (2 \sin x \cos x)^2 \right] \\
= 2 \sin x \cos x \left[ (\cos^4 x - 2 \sin^2 x \cos^2 x + \sin^4 x) - 4 \sin^2 x \cos^2 x \right] \\
= 2 \sin x \cos x \left[ 3 \cos^4 x - 6 \sin^2 x \cos^2 x + 3 \sin^4 x - 4 \sin^2 x \cos^2 x \right] \\
= 2 \sin x \cos x \left[ 3 \cos^4 x + 3 \sin^4 x - 10 \sin^2 x \cos^2 x \right] \\
= 6 \sin x \cos x + 6 \sin^3 x \cos x - 20 \sin^3 x \cos^3 x
\]

18. Using our new formula, \( \cos^4 x = \left( \frac{1}{2} (\cos 2x + 1) \right)^2 \) Now, our final answer needs to be in the first power of cosine, so we need to find a formula for \( \cos^2 2x \). For this, we substitute 2x everywhere there is an x and the formula translates to \( \cos^2 2x = \frac{1}{4} (\cos 4x + 1) \). Now we can write \( \cos^4 x \) in terms of the first power of cosine as follows.
\[
\cos^4 x = \left[\frac{1}{2}(\cos 2x + 1)\right]^2 \\
= \frac{1}{4}(\cos^2 2x + 2\cos 2x + 1) \\
= \frac{1}{4}\left(\frac{1}{2}(\cos 4x + 1) + 2\cos 2x + 1\right) \\
= \frac{1}{8}\cos 4x + \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4} \\
= \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}
\]

19. Using our new formula, \(\sin^4 x = \left[\frac{1}{2}(1 - \cos 2x)\right]^2\) Now, our final answer needs to be in the first power of cosine, so we need to find a formula for \(\cos^2 2x\). For this, we substitute \(2x\) everywhere there is an \(x\) and the formula translates to \(\cos^2 2x = \frac{1}{2}(\cos 4x + 1)\). Now we can write \(\sin^4 x\) in terms of the first power of cosine as follows.

\[
\sin^4 x = \left[\frac{1}{2}(1 - \cos 2x)\right]^2 \\
= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) \\
= \frac{1}{4}(1 - 2\cos 2x + \frac{1}{2}(\cos 4x + 1)) \\
= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x + \frac{1}{8} \\
= \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8}
\]

20. Answers: (a) First, we use both of our new formulas, then simplify:

\[
\sin^2 x \cos^2 2x = \frac{1}{2}(1 - \cos 2x)\left(\frac{1}{2}(\cos 4x + 1)\right) \\
= \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\left(\frac{1}{2}\cos 4x + \frac{1}{2}\right) \\
= \frac{1}{4}\cos 4x + \frac{1}{4} - \frac{1}{4}\cos 2x\cos 4x - \frac{1}{4}\cos 2x \\
= \frac{1}{4}(1 - \cos 2x + \cos 4x - \cos 2x\cos 4x)
\]

(b) For tangent, we use the identity \(\tan x = \frac{\sin x}{\cos x}\) and then substitute in our new formulas. \(\tan^4 2x = \frac{\sin^4 2x}{\cos^4 2x}\) Now, use the formulas we derived in \#18 and \#19.

\[
\tan^4 2x = \frac{\sin^4 2x}{\cos^4 2x} \\
= \frac{1}{8}\cos 8x - \frac{1}{2}\cos 4x + \frac{3}{8} \\
= \frac{1}{8}\cos 8x + \frac{1}{2}\cos 4x + \frac{3}{8} \\
= \frac{1}{8}\cos 8x - 4\cos 4x + 3 \\
= \frac{\cos 8x + 4\cos 4x + 3}{\cos 8x + 4\cos 4x + 3}
\]