Unit 3 – Test

1. If the terminal side is on $(-8,15)$, then the hypotenuse of this triangle would be 17 (by the Pythagorean Theorem, $c = \sqrt{(-8)^2+15^2}$). Therefore, $\sin x = \frac{15}{17}, \cos x = -\frac{8}{17}$, and $\tan x = -\frac{15}{8}$.
2. If $\sin a = \frac{\sqrt{5}}{3}$ and $\tan a < 0$, then a is in Quadrant II. Therefore $\sec a$ is negative. To find the third side, we need to do the Pythagorean Theorem.

$$\begin{aligned}\left(\sqrt{5}\right)^2 + b^2 &= 3^2\\ 5+b^2 &= 9\\ b^2 &= 4\\ b &= 2\end{aligned}$$

So $\sec a = -\frac{3}{2}$.

3. Factor top, cancel like terms, and use the Pythagorean Theorem Identity. Note that this simplification doesn't hold true for values of x that are $\frac{\pi}{4}+\frac{n\pi}{2}$, where n is a positive integer,, since the original expression is undefined for these values of x.

$$\begin{gathered}\frac{\cos^4 x - \sin^4 x}{\cos^2 x - \sin^2 x}\\ \frac{(\cos^2 x+\sin^2 x)(\cos^2 x - \sin^2 x)}{\cos^2 x - \sin^2 x}\\ \frac{\cos^2 x + \sin^2 x}{1}\\ \frac{1}{1}\\ 1\end{gathered}$$

4. Change secant and cosecant into terms of sine and cosine, then find a common denominator.

$$\begin{aligned}\frac{1+\sin x}{\cos x \sin x} &= \sec x(\csc x + 1)\\ &= \frac{1}{\cos x}\left(\frac{1}{\sin x}+1\right)\\ &= \frac{1}{\cos x}\left(\frac{1+\sin x}{\sin x}\right)\\ &= \frac{1+\sin x}{\cos x \sin x}\end{aligned}$$

5.

$$\begin{aligned}\sec\left(x+\frac{\pi}{2}\right)+2 &= 0\\ \sec\left(x+\frac{\pi}{2}\right) &= -2\\ \cos\left(x+\frac{\pi}{2}\right) &= -\frac{1}{2}\\ x+\frac{\pi}{2} &= \frac{2\pi}{3}, \frac{4\pi}{3}\\ x &= \frac{2\pi}{3}-\frac{\pi}{2}, \frac{4\pi}{3}-\frac{\pi}{2}\\ x &= \frac{\pi}{6}, \frac{5\pi}{6}\end{aligned}$$

6.

$$\begin{aligned}8\sin\left(\frac{x}{2}\right)-8&=0\\8\sin\frac{x}{2}&=8\\\sin\frac{x}{2}&=1\\\frac{x}{2}&=\frac{x}{2}\\x&=\pi\end{aligned}$$

7.

$$\begin{aligned}2\sin^2 x+\sin 2x&=0\\2\sin^2 x+2\sin x\cos x&=0\\2\sin x(\sin x+\cos x)&=0\end{aligned}$$

$$\begin{array}{lcl}\text{So, } 2\sin x=0 & \text{or} & \sin x+\cos x=0\\ 2\sin x=0 & & \sin x+\cos x=0\\ \sin x=0 & & \sin x=-\cos x\\ x=0,\pi & & x=\frac{3\pi}{4},\frac{7\pi}{4}\end{array}$$

8.

$$\begin{aligned}3\tan^2 2x&=1\\\tan^2 2x&=\frac{1}{3}\\\tan 2x&=\pm\frac{\sqrt{3}}{3}\\2x&=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6},\frac{19\pi}{6},\frac{23\pi}{6}\\x&=\frac{\pi}{12},\frac{5\pi}{12},\frac{7\pi}{12},\frac{11\pi}{12},\frac{13\pi}{12},\frac{17\pi}{12},\frac{19\pi}{12},\frac{23\pi}{12}\end{aligned}$$

9.

$$\begin{aligned}1-\sin x&=\sqrt{3}\sin x\\1&=\sin x+\sqrt{3}\sin x\\1&=\sin x\left(1+\sqrt{3}\right)\\\frac{1}{1+\sqrt{3}}&=\sin x\end{aligned}$$

$\sin^{-1}\left(\frac{1}{1+\sqrt{3}}\right)=x$ or $x=.3747$ radians and $x=2.7669$ radians

10. Because this is $\cos 3x$, you will need to divide by 3 at the very end to get the final answer. This is why we

went beyond the limit of 2π when finding $3x$.

$$\begin{aligned} 2\cos 3x - 1 &= 0 \\ 2\cos 3x &= 1 \\ \cos 3x &= \frac{1}{2} \\ 3x &= \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3} \\ x &= \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9} \end{aligned}$$

11. Rewrite the equation in terms of tan by using the Pythagorean identity, $1 + \tan^2\theta = \sec^2\theta$.

$$\begin{aligned} 2\sec^2 x - \tan^4 x &= 3 \\ 2(1 + \tan^2 x) - \tan^4 x &= 3 \\ 2 + 2\tan^2 x - \tan^4 x &= 3 \\ \tan^4 x - 2\tan^2 x + 1 &= 0 \\ (\tan^2 x - 1)(\tan^2 x - 1) &= 0 \end{aligned}$$

Because these factors are the same, we only need to solve one for x.

$$\begin{aligned} \tan^2 x - 1 &= 0 \\ \tan^2 x &= 1 \\ \tan x &= \pm 1 \\ x &= \frac{\pi}{4} + \pi k \text{ and } \frac{3\pi}{4} + \pi k \end{aligned}$$

Where k is any integer.

12. Use the half angle formula with 315°.

$$\begin{aligned} \cos 157.5^\circ &= \cos \frac{315^\circ}{2} \\ &= -\sqrt{\frac{1 + \cos 315^\circ}{2}} \\ &= -\sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} \\ &= -\sqrt{\frac{2 + \sqrt{2}}{4}} \\ &= -\frac{\sqrt{2 + \sqrt{2}}}{2} \end{aligned}$$

13. Use the sine sum formula.

$$\begin{aligned} \sin \frac{13\pi}{12} &= \sin\left(\frac{10\pi}{12} + \frac{3\pi}{12}\right) \\ &= \sin\left(\frac{5\pi}{6} + \frac{\pi}{4}\right) \\ &= \sin \frac{5\pi}{6} \cos \frac{\pi}{4} + \cos \frac{5\pi}{6} \sin \frac{\pi}{4} \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + (-\frac{\sqrt{3}}{2}) \cdot \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2} - \sqrt{6}}{4} \end{aligned}$$

14.

$$
\begin{aligned}
4(\cos 5x+\cos 9x) &= 4\left[2\cos\left(\frac{5x+9x}{2}\right)\cos\left(\frac{5x-9x}{2}\right)\right] \\
&= 8\cos 7x\cos(-2x) \\
&= 8\cos 7x\cos 2x
\end{aligned}
$$

15.

$$
\begin{gathered}
\cos(x-y)\cos y-\sin(x-y)\sin y \\
\cos y(\cos x\cos y+\sin x\sin y)-\sin y\,(\sin x\cos y-\cos x\sin y) \\
\cos x\cos^2 y+\sin x\sin y\cos y-\sin x\sin y\cos y+\cos x\sin^2 y \\
\cos x\cos^2 y+\cos x\sin^2 y \\
\cos x(\cos^2 y+\sin^2 y) \\
\cos x
\end{gathered}
$$

16. Use the sine and cosine sum formulas.

$$
\begin{gathered}
\sin\left(\frac{4\pi}{3}-x\right)+\cos\left(x+\frac{5\pi}{6}\right) \\
\sin\frac{4\pi}{3}\cos x-\cos\frac{4\pi}{3}\sin x+\cos x\cos\frac{5\pi}{6}-\sin x\sin\frac{5\pi}{6} \\
-\frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x-\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x \\
-\sqrt{3}\cos x
\end{gathered}
$$

17. Use the sine sum formula as well as the double angle formula.

$$
\begin{aligned}
\sin 6x &= \sin(4x+2x) \\
&= \sin 4x\cos 2x+\cos 4x\sin 2x \\
&= \sin(2x+2x)\cos 2x+\cos(2x+2x)\sin 2x \\
&= \cos 2x\,(\sin 2x\cos 2x+\cos 2x\sin 2x)+\sin 2x(\cos 2x\cos 2x-\sin 2x\,\sin 2x) \\
&= 2\sin 2x\cos^2 2x+\sin 2x\cos^2 2x-\sin^3 2x \\
&= 3\sin 2x\,\cos^2 2x-\sin^3 2x \\
&= \sin 2x(3\cos^2 2x-\sin^2 2x) \\
&= 2\sin x\cos x[3(\cos^2 x-\sin^2 x)^2-(2\sin x\cos x)^2 \\
&= 2\sin x\cos x[3(\cos^4 x-2\sin^2 x\cos^2 x+\sin^4 x)-4\,\sin^2 x\cos^2 x] \\
&= 2\sin x\cos x[3\cos^4 x-6\sin^2 x\cos^2 x+3\sin^4 x-4\,\sin^2 x\cos^2 x] \\
&= 2\sin x\cos x[3\cos^4 x+3\sin^4 x-10\sin^2 x\cos^2 x] \\
&= 6\sin x\cos^5 x+6\sin^5 x\cos x-20\sin^3 x\cos^3 x
\end{aligned}
$$

18. Using our new formula, $\cos^4 x=\left[\frac{1}{2}(\cos 2x+1)\right]^2$ Now, our final answer needs to be in the first power of cosine, so we need to find a formula for $\cos^2 2x$. For this, we substitute $2x$ everywhere there is an x and the formula translates to $\cos^2 2x=\frac{1}{2}(\cos 4x+1)$. Now we can write $\cos^4 x$ in terms of the first power of cosine as follows.

$$\begin{aligned}\cos^4 x &= [\frac{1}{2}(\cos 2x+1)]^2\\ &= \frac{1}{4}(\cos^2 2x+2\cos 2x+1)\\ &= \frac{1}{4}(\frac{1}{2}(\cos 4x+1)+2\cos 2x+1)\\ &= \frac{1}{8}\cos 4x+\frac{1}{8}+\frac{1}{2}\cos 2x+\frac{1}{4}\\ &= \frac{1}{8}\cos 4x+\frac{1}{2}\cos 2x+\frac{3}{8}\end{aligned}$$

19. Using our new formula, $\sin^4 x = \left[\frac{1}{2}(1-\cos 2x)\right]^2$ Now, our final answer needs to be in the first power of cosine, so we need to find a formula for $\cos^2 2x$. For this, we substitute $2x$ everywhere there is an x and the formula translates to $\cos^2 2x = \frac{1}{2}(\cos 4x+1)$. Now we can write $\sin^4 x$ in terms of the first power of cosine as follows.

$$\begin{aligned}\sin^4 x &= [\frac{1}{2}(1-\cos 2x)]^2\\ &= \frac{1}{4}(1-2\cos 2x+\cos^2 2x)\\ &= \frac{1}{4}(1-2\cos 2x+\frac{1}{2}(\cos 4x+1))\\ &= \frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x+\frac{1}{8}\\ &= \frac{1}{8}\cos 4x-\frac{1}{2}\cos 2x+\frac{3}{8}\end{aligned}$$

20. Answers: (a) First, we use both of our new formulas, then simplify:

$$\begin{aligned}\sin^2 x\cos^2 2x &= \frac{1}{2}(1-\cos 2x)\frac{1}{2}(\cos 4x+1)\\ &= \left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)\left(\frac{1}{2}\cos 4x+\frac{1}{2}\right)\\ &= \frac{1}{4}\cos 4x+\frac{1}{4}-\frac{1}{4}\cos 2x\cos 4x-\frac{1}{4}\cos 2x\\ &= \frac{1}{4}(1-\cos 2x+\cos 4x-\cos 2x\cos 4x)\end{aligned}$$

(b) For tangent, we use the identity $\tan x = \frac{\sin x}{\cos x}$ and then substitute in our new formulas. $\tan^4 2x = \frac{\sin^4 2x}{\cos^4 2x} \rightarrow$ Now, use the formulas we derived in #18 and #19.

$$\begin{aligned}\tan^4 2x &= \frac{\sin^4 2x}{\cos^4 2x}\\ &= \frac{\frac{1}{8}\cos 8x-\frac{1}{2}\cos 4x+\frac{3}{8}}{\frac{1}{8}\cos 8x+\frac{1}{2}\cos 4x+\frac{3}{8}}\\ &= \frac{\cos 8x-4\cos 4x+3}{\cos 8x+4\cos 4x+3}\end{aligned}$$