Solving Equations: Lesson Summary with Examples

Properties of Equality

Remember, the properties of equality demonstrate the operation performed to each side of the equation.

<table>
<thead>
<tr>
<th>Addition Property of Equality</th>
<th>Subtraction Property of Equality</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( a = b ), then ( a + c = b + c ).</td>
<td>If ( a = b ), then ( a - c = b - c ).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Multiplication Property of Equality</th>
<th>Division Property of Equality</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( a = b ), then ( a \cdot c = b \cdot c ).</td>
<td>If ( a = b ), then ( \frac{a}{c} = \frac{b}{c} ).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symmetric Property of Equality</th>
<th>Distributive Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( a = b ), then ( b = a ).</td>
<td>( a(b + c) = ab + ac ) and ( a(b - c) = ab - ac ) ( (b + c)a = ab + ac ) and ( (b - c)a = ab - ac )</td>
</tr>
</tbody>
</table>

Subtraction Property of Equality

\[
\begin{align*}
5z - 6 &= 3z + 2 \\
5z - 6 - 3z &= 3z + 2 - 3z \\
2z - 6 &= 2 - 3z
\end{align*}
\]

Multiplication Property of Equality

\[
\begin{align*}
\frac{x}{7} &= -6 \\
7 \cdot \frac{x}{7} &= 7 \cdot (-6) \\
x &= -42
\end{align*}
\]

Division Property of Equality

\[
\begin{align*}
5(x + 3) &= -10 \\
\frac{5(x + 3)}{5} &= -10 \\
x + 3 &= -2
\end{align*}
\]

Addition Property of Equality

\[
\begin{align*}
x &= -16 \\
x + (-16) &= 8 \\
x + (-16) + (-16) &= 8 + (-16)
\end{align*}
\]

Distributive Property

The Distributive Property can be used to simplify expressions.

\[
\begin{align*}
-3(2x - 1) &= -6x + 3 \\
(7 + d)2 &= 14 + 2d
\end{align*}
\]
Justifying Solutions of Equations

The properties of equality can be used to justify each step used to solve an equation.

<table>
<thead>
<tr>
<th>Algebraic Steps</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3(x + 8) = -21$</td>
<td><em>Original equation</em></td>
</tr>
<tr>
<td>$3x + 24 = -21$</td>
<td>Distributive Property</td>
</tr>
<tr>
<td>$3x + 24 - 24 = -21 - 24$</td>
<td>Subtraction Property of Equality</td>
</tr>
<tr>
<td>$3x = -45$</td>
<td>Simplify</td>
</tr>
<tr>
<td>$\frac{3x}{3} = \frac{-45}{3}$</td>
<td>Division Property of Equality</td>
</tr>
<tr>
<td>$x = -15$</td>
<td>Simplify</td>
</tr>
</tbody>
</table>

Modeling Mathematical Problems

Equations can be created to model mathematical problems, and then used to solve them.

The sum of three consecutive integers is 66.

1st number: $x$  
2nd number: $x + 1$  
3rd number: $x + 2$

$21 + 22 + 23 = 66 \checkmark$
Solving Formulas and Literal Equations: Lesson Summary with Examples

A literal equation involves two or more variables. Remember, there will be restrictions if there are variables in the denominator. To find the restrictions find the value of the variable for which the denominator is equal to zero.

**Rewriting Literal Equations**

The equation \( d = rt \) can be solved for \( r \) or \( t \).

**Solve for \( r \).**

\[
\begin{align*}
  r &= \frac{d}{t} \\
  d &= rt \\
  t &= \frac{d}{r}
\end{align*}
\]

**Solve for \( t \).**

\[
\begin{align*}
  t &= \frac{d}{r} \\
  d &= rt \\
  r &= \frac{d}{t}
\end{align*}
\]

**Solve the literal equation for \( y \).**

\[
\begin{align*}
  3x + y &= 10 \\
  3x + y - 3x &= 10 - 3x \\
  y &= 10 - 3x
\end{align*}
\]

**Solve the literal equation for \( x \).**

\[
\begin{align*}
  3x + y &= 10 \\
  3x + y - y &= 10 - y \\
  3x &= 10 - y \\
  \frac{3x}{3} &= \frac{10 - y}{3} \\
  x &= \frac{10 - y}{3}
\end{align*}
\]

**Finding Restrictions**

Solve the equation for \( x \).

1. Use the Distributive Property to simplify. \( 4x + xy = 6 \)
   \[
   x(4 + y) = 6
   \]
2. Divide each side by \( 4 + y \). \[
   \frac{x(4 + y)}{4 + y} = \frac{6}{4 + y}
   \]
3. Simplify. \[
   x = \frac{6}{4 + y}
   \]
4. State any restrictions. \( y \neq -4 \)

Since it is not possible to divide by zero, there are restrictions on the variable when variables are in the denominator. To find the restrictions set the denominator equal to zero and solve for the variable.
Rewriting Formulas

Solve \( C = \frac{5}{9}(F - 32) \) for \( F \).

Two student explanations:

<table>
<thead>
<tr>
<th>Vivian's Work</th>
<th>Chander's Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>I'll start by distributing. ( C = \frac{5}{9}(F - 32) )</td>
<td>I'll start by multiplying each side by the reciprocal. ( C = \frac{5}{9}(F - 32) )</td>
</tr>
<tr>
<td>Then I can add to each side. ( C + \frac{160}{9} = \frac{5}{9}F - \frac{160}{9} + \frac{160}{9} )</td>
<td>And then I can add to each side. ( \frac{9}{5}C + 32 = F - 32 + 32 )</td>
</tr>
<tr>
<td>And finally multiply each side by the reciprocal and simplify. ( \frac{9}{5} \left( C + \frac{160}{9} \right) = \frac{5}{9}F \cdot \frac{9}{5} )</td>
<td>( \frac{9}{5}C + 32 = F )</td>
</tr>
</tbody>
</table>
Solving Compound Inequalities: Lesson Summary with Examples

A compound inequality is two inequalities connected by the word *and* or the word *or*. The intersection of a compound inequality is the solution that is common to both inequalities in a compound inequality. The union of a compound inequality contains the solution to either inequality, but not necessarily both.

To solve a compound inequality

1. Write two inequalities.
2. Isolate the variable by using inverse operations for each inequality.
3. Solve each inequality.

Possible solutions to a compound inequality include two inequalities, one inequality, all real numbers, or no solution.

When graphing a solution, be sure to consider whether the solution represents an intersection or a union.

### Compound Inequalities

The solution of a compound inequality can be represented as an intersection or a union and graphed on a number line.

The solution of the compound inequality containing the word *and* must satisfy both inequalities.

Graph $x \geq -3$ and $x < 2$.

The solution of a compound inequality containing the word *or* must satisfy only one of the inequalities.

Graph $x < -3$ or $x \geq 2$.

### Solving Compound Inequalities: Intersection

When solving a *compound* inequality such as $-8 < x - 3 < 1$, it's important to remember that since we are solving two inequalities with respect to one expression, any operations used to solve one inequality will also be used to solve the other.

Solve and graph the solution for the compound inequality.

\[
-8 < x - 3 < 1 \\
-8 + 3 < x - 3 + 3 < 1 + 3 \\
-5 < x < 4
\]
Solving Compound Inequalities: Union

Each inequality of a compound inequality representing a union can be solved independently.

Solve and graph $3x + 10 \leq 25$ or $-2(x - 2) < -10$.

1. Solve each inequality.

<table>
<thead>
<tr>
<th>$3x + 10 \leq 25$</th>
<th>$-2(x - 2) &lt; -10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x + 10 - 10 \leq 25 - 10$</td>
<td>$-2x + 4 &lt; -10$</td>
</tr>
<tr>
<td>$3x \leq 15$</td>
<td>$-2x + 4 - 4 &lt; -10 - 4$</td>
</tr>
<tr>
<td>$\frac{3x}{3} \leq \frac{15}{3}$</td>
<td>$-2x &lt; -14$</td>
</tr>
<tr>
<td>$x \leq 5$</td>
<td>$-2x &gt; -14 - 4$</td>
</tr>
<tr>
<td>$x \leq 5$</td>
<td>$x &gt; \frac{-14}{2}$</td>
</tr>
<tr>
<td>$x \leq 5$</td>
<td>$x &gt; -7$</td>
</tr>
</tbody>
</table>

$x \leq 5$ or $x > 7$

2. Graph the solution on a number line.

![Number Line Graph](image-url)
Solving Inequalities: Lesson Summary with Examples

When graphing the solution for an inequality:

- An open circle is used for < and > to indicate the number is not included in the solution.
- A closed circle is used for ≤ and ≥ to indicate the number is included in the solution.

Key Point

The inequality symbol stays the same when:
- Adding to or subtracting from each side
- Multiplying or dividing each side by a positive number

The inequality symbol flips when:
- Multiplying or dividing each side by a negative number

Inequalities

Write an inequality for each statement.

- A number $t$ is no more than $-13$.  
  $$t \leq -13$$
- Fifty is fewer than a number $q$.  
  $$50 < q$$
- The sum of a number $h$ and 9 is at least 22.  
  $$h + 9 \geq 22$$
- The product of 4 and a number $y$ is more than 0.  
  $$4y > 0$$
- Four is at most the difference of $-5$ and some number $k$.  
  $$4 \leq -5 - k$$
Solving Using Addition and Subtraction

Solve \( x - 5 < -1 \).

Addition Property of Inequality \[ x - 5 + 5 < -1 + 5 \]

Simplify. \[ x < 4 \]

Graph the solution.

Solving Using Multiplication and Division

Solve \( \frac{w}{-7} \geq -1 \).

Multiplication Property of Inequality \[ -7 \cdot \frac{w}{-7} \geq -1(-7) \]

Simplify. \[ w \leq 7 \]

Graph the solution.

Solving Multi-Step Inequalities

Solve \( \frac{-b+2}{3} < 7 \).

Multiply each side by 3. \[ 3 \cdot \frac{-b+2}{3} < 7 \cdot 3 \]

Simplify. \[ -b + 2 < 21 \]

Subtract 2 from each side. \[ -b + 2 - 2 < 21 - 2 \]

Simplify. \[ -b < 19 \]

Divide each side by -1. \[ \frac{-b}{-1} < \frac{19}{-1} \]

Simplify. \[ b > -19 \]

Graph the solution.
Solving Inequalities with Variables on Both Sides

Solve \( 2(x + 5) - 9 \geq 3(x + 1) \)

\[
\begin{align*}
2(x + 5) - 9 & \geq 3(x + 1) \\
2x + 10 - 9 & \geq 3x + 3 \\
2x + 1 & \geq 3x + 3 \\
2x + 1 - 3x & \geq 3x + 3 - 3x \\
-x + 1 & \geq 3 \\
-x + 1 - 1 & \geq 3 - 1 \\
-x & \geq 2 \\
\frac{-x}{-1} & \leq \frac{2}{-1} \\
x & \leq -2
\end{align*}
\]
Solving Absolute Value Equations and Inequalities: Lesson Summary with Examples

To solve an absolute value equation
1. Isolate the absolute value expression, if necessary.
2. Write two equations.
3. Solve each equation.

An extraneous solution is a solution that does not satisfy the original equation. Absolute value inequalities can be solved using compound inequalities.

To solve an absolute value inequality
1. Isolate the absolute value expression, if necessary.
2. Write a compound inequality.
3. Solve each inequality.

Absolute Value Expressions
The absolute value of a number is its distance from zero on a number line. Absolute value expressions can be evaluated by substituting a value for a variable in the expression.

Find $|t - 8| + 5$ when $t = -9$.

$|-9 - 8| + 5$

$|-17| + 5$

$17 + 5$

$22$
Absolute Value Equations

Solve $|x + 2| = 5$.

Using if $|x| = n$, then $x = n$ or $x = -n$, write two equations, and then solve each equation.

$x + 2 = 5$ or $x + 2 = -5$

$x + 2 - 2 = 5 - 2$ or $x + 2 - 2 = -5 - 2$

$x = 3$ or $x = -7$

The solutions are $x = 3$ or $x = -7$.

To solve absolute value equations, like $|p + 3| + 1 = 7$, it is necessary to first isolate the absolute value expression. Click each step to see how to solve this type of equation.

Solve $|p + 3| + 1 = 7$.

1. Isolate the absolute value expression.

   Subtract 1 from each side.

   $|p + 3| + 1 = 7$
   $|p + 3| + 1 - 1 = 7 - 1$
   $|p + 3| = 6$

2. Write two equations.

   $|p + 3| = 6$
   $p + 3 = 6$ or $p + 3 = -6$

3. Solve each equation.

   $p + 3 = 6$ or $p + 3 = -6$
   $p + 3 - 3 = 6 - 3$ or $p + 3 - 3 = -6 - 3$
   $p = 3$ or $p = -9$

   The solutions are $p = 3$ or $p = -9$.

4. Check the solutions in the original equation.

   Always check the solutions in the original equation.

   Check: $p = 3$
   $|3 + 3| + 1 = 7$
   $|6| + 1 = 7$
   $6 + 1 = 7$
   $7 = 7$ ✓

   Check: $p = -9$
   $|-9 + 3| + 1 = 7$
   $|6| + 1 = 7$
   $6 + 1 = 7$
   $7 = 7$ ✓
Extraneous Solutions

Extraneous solutions may occur when solving absolute value equations such as $|2x + 5| = 4x - 1$.

Solve $|2x + 5| = 4x - 1$.

$$
\begin{align*}
2x + 5 &= 4x - 1 \\
2x + 5 - 4x &= 4x - 1 - 4x \\
-2x + 5 &= -1 \\
-2x + 5 - 5 &= -1 - 5 \\
-2x &= -6 \\
\frac{-2x}{-2} &= \frac{-6}{-2} \\
x &= 3.
\end{align*}
$$

Check: $x = 3$

$$
|2(3) + 5| = 4(3) - 1
$$

$$
|11| = 11 \\
11 = 11 \checkmark
$$

$$
\begin{align*}
2x + 5 &= -(4x - 1) \\
2x + 5 &= -4x + 1 \\
2x + 5 + 4x &= -4x + 1 + 4x \\
6x + 5 &= 1 \\
6x + 5 - 5 &= 1 - 5 \\
6x &= -4 \\
\frac{6x}{6} &= \frac{-4}{6} \\
x &= -\frac{2}{3}.
\end{align*}
$$

Check: $x = -\frac{2}{3}$

$$
\left|2\left(-\frac{2}{3}\right) + 5\right| = 4\left(-\frac{2}{3}\right) - 1
$$

$$
\left|\frac{11}{3}\right| = -\frac{11}{3} \\
\frac{11}{3} \neq -\frac{11}{3}
$$

The only solution to the equation is $x = 3$. 
Absolute Value Inequalities Involving $< \text{ or } \leq$

Solve $|2x + 5| < 15$.

Using if $|x| < n$, then $-n < x < n$, write a compound inequality, and then solve each inequality.

$-15 < 2x + 5 < 15$

$-15 - 5 < 2x + 5 - 5 < 15 - 5$

$-20 < 2x < 10$

$-\frac{20}{2} < \frac{2x}{2} < \frac{10}{2}$

$-10 < x < 5$

Graph the solution.

Carolina Tiger Rescue is a nonprofit organization that provides a safe home for rescued wildcats such as lions, tigers, ocelots and cougars. The average weight of a male cougar is 165 pounds, give or take 55 pounds. How much could a male cougar weigh?

We can write an absolute value inequality to represent this situation.

The absolute value of the difference in the cougar’s actual weight and 165 pounds is at most 55 pounds.

Let $w$ represent the actual weight of a male cougar.

$|w - 165| \leq 55$

Solve the absolute value inequality to find the male cougar’s weight range.

$|w - 165| \leq 55$

$-55 \leq w - 165 \leq 55$

$-55 + 165 \leq w - 165 + 165 \leq 55 + 165$

$110 \leq w \leq 220$

Absolute Value Inequalities Involving $> \text{ or } \geq$

Solve $|7x| \geq 42$.

Using if $|x| \geq n$, then $x \leq -n \text{ or } x \geq n$, write a compound inequality, and then solve each inequality.

$7x \leq -42 \text{ or } 7x \geq 42$

$\frac{7x}{7} \leq -\frac{42}{7} \text{ or } \frac{7x}{7} \geq \frac{42}{7}$

$x \leq -6 \text{ or } x \geq 6$
Functions and Relations: Lesson Summary with Examples

A relation is a set of ordered pairs. The domain of a relation is the set of inputs, or the x-coordinates of the ordered pairs. The range of a relation is the set of outputs, or the y-coordinates of the ordered pairs. A function is a special type of relation where each input has exactly one output.

You can look at a relation in many forms (ordered pairs, table, mapping diagram, graph) and determine if it is a function. It may be helpful to use a mapping diagram or the vertical line test to help you determine if a relation is a function.

If a relation maps an element \( a \) in the domain to element \( b \) in the range, the inverse relation "reverses" the mapping and maps \( b \) back to \( a \). This means if \((a, b)\) is an ordered pair in a relation, then \((b, a)\) is an ordered pair of its inverse. The inverse of a relation or the inverse of an equation are both a reflection of the original relation or equation over the line \( y = x \).

To find the inverse of a relation represented as an equation

1. Interchange the \( x \) and \( y \).
2. Solve for \( y \).

Domain and Range of a Relation

A relation is a set of ordered pairs and can be represented multiple ways.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

\((-3, 2), (-1, 0), (1, -2), (1, 3)\)

Relation: \{(-3, 2), (-1, 0), (1, -2), (1, 3)\}

Domain: \{-3, -1, 1\}

Range: \{-2, 0, 2, 3\}

The domain of a relation is the set of inputs, or the x-coordinates of the ordered pairs. The range of a relation is the set of outputs, or the y-coordinates of the ordered pairs.
Identifying Functions

A function is a special type of a relation where each input has *exactly one* output.

This relation is a function. Each value in the domain (the input) is mapped to exactly one value in the range (the output).

This relation is not a function. Both values in the domain are mapped to more than one value in the range. The number 1 maps to both 3 and 5. The number 2 maps to both –1 and 6.

This relation is a function. While two different numbers in the domain, 2 and 3, map to 5, it still meets the definition that each input maps to exactly one output.

**Key Point**

If a value in the domain repeats, then the relation is not a function.

The Vertical Line Test

One way to determine if the relation is a function is to use the vertical line test. This test says that if any vertical line passes through one *and only one* point on the graph at a time, then the graph is a function.

This graph is a function since it passes the vertical line test.
Inverse of a Relation

The inverse of a relation reverses the mapping of the domain to the range. So if a relation maps a domain value \( a \) to a range value \( b \), the inverse relation “reverses” the mapping and maps \( b \) back to \( a \). This means if \((a, b)\) is an ordered pair in a relation, then \((b, a)\) is an ordered pair in the inverse of the relation.

<table>
<thead>
<tr>
<th>Relation</th>
<th>Inverse</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-3, 6))</td>
<td>((6, -3))</td>
</tr>
<tr>
<td>((4, 0))</td>
<td>((0, 4))</td>
</tr>
<tr>
<td>((0, -2))</td>
<td>((-2, 0))</td>
</tr>
<tr>
<td>((a, b))</td>
<td>((b, a))</td>
</tr>
</tbody>
</table>

The graph of a relation and its inverse are reflections across the line \( y = x \).

relation: \{\((-3, 3), (-3, 0), (4, 5)\)\}

inverse: \{\((3, -3), (0, -3), (5, 4)\)\}

Inverse of an Equation

To find the inverse of an equation, interchange the \( x \) and \( y \) variables and solve for \( y \).

Find the inverse of \( y = 2x + 4 \).

Interchange the \( x \) and \( y \).

\[ x = 2y + 4 \]

Solve for \( y \).

\[ x - 4 = 2y + 4 - 4 \]
\[ x - 4 = 2y \]
\[ \frac{x - 4}{2} = \frac{2y}{2} \]
\[ y = \frac{1}{2}x - 2 \]
Evaluating Functions: Lesson Summary with Examples

A function rule describes an output variable in terms of an input variable. Function notation is a way to name a function defined by an equation, for example \( f(x) = 3x - 5 \). A function can be evaluated for a single value, an expression, or for a given domain. The notation \( f^{-1}(x) \) is used to denote the inverse of a function \( f(x) \).

Functions can be combined using the arithmetic operations: addition, subtraction, multiplication, and division. When the function is representing a real world situation, combining functions using arithmetic operations can give us a new function to help us solve different problems.

Defining a Function Rule and Function notation

\[ f(x) = x - 2, \] is a function rule written using function notation. Function notation is a way to name a function defined by an equation.

- \( f \) is the name of the function or “rule”
- \( x \) is the input value
- \( f(x) \) is the output when the rule \( f \) is applied to \( x \)
- \( f(x) \) is read as “\( f \) of \( x \)”

Evaluating Functions

Evaluate \( f(x) = 3x + 4 \) for \( x = -3 \).

Substitute \(-3\) for \( x \) in the function.  
\[ f(-3) = 3(-3) + 4 \]
Simplify.  
\[ f(-3) = -9 + 4 \]
\[ f(-3) = -5 \]

Evaluate \( f(x) = 4x - 5 \) for \( f(t + 2) \).

Substitute \( t + 2 \) for \( x \) in the function.  
\[ f(t + 2) = 4(t + 2) - 5 \]
Simplify.  
\[ f(t + 2) = 4t + 8 - 5 \]
\[ f(t + 2) = 4t + 3 \]

Find \( f(5) \) if \( f(x) = x^2 + 2x + 3 \)

\[ f(5) = 5^2 + 2(5) + 3 \]
\[ f(5) = 25 + 10 + 3 \]
\[ f(5) = 38 \]

Evaluating Functions for a Given Domain

Find the range of \( f(x) = 4x - 2 \) for the domain \([-2, 0, 4] \).

\[ f(-2) = 4(-2) - 2 \]
\[ f(0) = 4(0) - 2 \]
\[ f(4) = 4(4) - 2 \]
\[ f(-2) = -8 - 2 \]
\[ f(0) = 0 - 2 \]
\[ f(4) = 16 - 2 \]
\[ f(-2) = -10 \]
\[ f(0) = -2 \]
\[ f(4) = 14 \]

range: \([-10, -2, 14] \)
Create a New Function

Two functions can be combined using all four basic operations.

\[
(f + g)(x) = f(x) + g(x) \quad (f - g)(x) = f(x) - g(x) \quad (f \cdot g)(x) = f(x) \cdot g(x) \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}
\]

If \( f(x) = 2x + 3 \) and \( g(x) = x \)

\[
(f + g)(4) = f(4) + g(4) \quad (f - g)(4) = f(4) - g(4) \quad (f \cdot g)(4) = f(4) \cdot g(4) \quad \left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)}
\]

\[
(f + g)(4) = 2(4) + 3 + 4 \quad (f - g)(4) = 2(4) + 3 - 4 \quad (f \cdot g)(4) = (2(4) + 3)(4) \quad \left(\frac{f}{g}\right)(4) = \frac{2(4) + 3}{4}
\]

\[
(f + g)(4) = 8 + 3 + 4 \quad (f - g)(4) = 8 + 3 - 4 \quad (f \cdot g)(4) = (8 + 3)(4) \quad \left(\frac{f}{g}\right)(4) = \frac{8 + 3}{4}
\]

\[
(f + g)(4) = 15 \quad (f - g)(4) = 7 \quad (f \cdot g)(4) = 44 \quad \left(\frac{f}{g}\right)(4) = \frac{11}{4}
\]

Finding the Inverse in Function Notation

Find the inverse of \( f(x) = \frac{2x-3}{3} \).

1. Rewrite the equation using \( y \).
   \[
y = \frac{2x-3}{3}
   \]

2. Interchange the \( x \) and \( y \) variables.
   \[
x = \frac{2y-3}{3}
   \]

3. Solve the new equation for \( y \).
   \[
   3x = 2y - 3 \\
   3x + 3 = 2y \\
   y = \frac{3x+3}{2}
   \]

4. Rewrite the equation using \( f^{-1}(x) \).
   \[
f^{-1}(x) = \frac{3x+3}{2}
   \]
Is the Inverse a Function?

If \( f(x) = 3x - 2 \) for the domain \(-2, -1, 0, 1, 2\), is the inverse a function?

1. Evaluate the function for each of the given domain values.

<table>
<thead>
<tr>
<th>( f(x) = 3x - 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(-2) = 3(-2) - 2 )</td>
</tr>
<tr>
<td>( f(-2) = -8 )</td>
</tr>
</tbody>
</table>

2. Write the ordered pairs for the function.

\((-2, -8), (-1, -5), (0, -2), (1, 1), (2, 4)\)

3. Find the inverse points.

\((-2, -8) \rightarrow (-8, -2), (-1, -5) \rightarrow (-5, -1), (0, -2) \rightarrow (-2, 0), (1, 1) \rightarrow (1, 1), (2, 4) \rightarrow (4, 2)\)

Remember if \((a, b)\) is an ordered pair in a relation, then \((b, a)\) is an ordered pair of its inverse.

4. Graph the inverse and identify if the inverse is a function.

The inverse of the given function, is itself a function, as it passes the vertical line test.
Arithmetic Sequences: Lesson Summary with Examples

A sequence is a set of numbers in a specific order, with each number being a term. An arithmetic sequence is a sequence that increases (adding) or decreases (subtracting) by a constant value. The constant value in an arithmetic sequence is the common difference.

Recursive formulas are used to determine the next term of a sequence from one or more of the preceding terms.

Explicit formulas are used to determine ANY term in a sequence.

Formulas for Arithmetic Sequences

Recursive Formula for an Arithmetic Sequence
Given a starting term and a common difference, \( d \), each term in an arithmetic sequence can be found as follows:
\[
\text{NEXT} = \text{NOW} + d
\]

Recursive Formula in Function Notation for an Arithmetic Sequence
Each term in an arithmetic sequence can be found as follows:
\[
f(n + 1) = f(n) + d
\]
where
- \( n \) is the number of the term
- \( f(n) \) is the \( n^{\text{th}} \) term in the sequence
- \( f(n + 1) \) is the \((n + 1)^{\text{th}}\) term in the sequence
- \( d \) is the common difference
- \( f(1) \) is the starting term

Explicit Formula
Any term in an arithmetic sequence can be found as follows:
\[
A(n) = A(1) + (n - 1)d
\]
where
- \( n \) is the number of the term
- \( A(n) \) is the \( n^{\text{th}} \) term in the sequence
- \( A(1) \) is the first term in the sequence
- \( d \) is the common difference
What is a Sequence?

A sequence is a set of numbers in a specific order. The order can be determined by any mathematical operation.

1, 4, 9, 16, ...

2, 4, 6, 8, 10, ...

6, 24, 96, 384, ...

Each number in the sequence is a term.

\[
\begin{array}{c}
\text{term} \\
\downarrow \\
2, 4, 6, 8, 10, ...
\end{array}
\]

\[
\begin{array}{c}
\text{term} \\
\uparrow \\
\text{term}
\end{array}
\]

Find the next two terms in each sequence.

25, 35, 45, 55

\[+10 +10 +10\]

\[55 + 10 = 65\]

\[65 + 10 = 75\]

55, 75

-2, 4, -8, 16

\[-2 \times (-2) \times (-2)\]

\[16 \times (-2) = -32\]

\[-32 \times (-2) = 64\]

-32, 64

1, 4, 9, 16, 25, 36

\[1^2 2^2 3^2 4^2 5^2 6^2\]

25, 36
Arithmetic Sequences

An arithmetic sequence is a sequence that increases or decreases by a constant value. Each new term is found by adding or subtracting this constant value from the previous term. The constant value in an arithmetic sequence is the common difference.

You can see that there is a pattern of subtracting 5 (or adding \(-5\)) from each term. So the common difference is \(-5\).

![Key Point]

When you are given a sequence, you can determine whether or not it is an arithmetic sequence by looking at the pattern. If the sequence has a common difference, \(d\), then it is an arithmetic sequence.

Using a Recursive Formula

A recursive formula is used to determine the next term of a sequence from one or more of the preceding terms.

![Recursive Formula for an Arithmetic Sequence]

Given a starting term and a common difference, \(d\), each term in an arithmetic sequence can be found as follows:

\[
NEXT = NOW + d
\]

Write the first 5 terms in this sequence.

\(NEXT = NOW + 3\), starting at 10

\[
\begin{align*}
Next &= 10 + 3 = 13 \\
Next &= 13 + 3 = 16 \\
Next &= 16 + 3 = 19 \\
Next &= 19 + 3 = 22
\end{align*}
\]

\[10, 13, 16, 19, 22\]
Function Notation

Write the first 4 terms of the arithmetic sequence.

\[ f(n + 1) = f(n) + 11, \quad f(1) = -10 \]

\[
\begin{align*}
    f(1) &= -10 \\
    f(2) &= f(1) + 11 \\
    f(3) &= f(2) + 11 \\
    f(4) &= f(3) + 11 \\
    f(2) &= -10 + 11 \\
    f(3) &= 1 + 11 \\
    f(4) &= 12 + 11 \\
    f(2) &= 1 \\
    f(3) &= 12 \\
    f(4) &= 23
\end{align*}
\]

\[-10, 1, 12, 23\]

Using an Explicit Formula

An explicit formula is used to determine any term in a sequence.

\[
\text{Any term in an arithmetic sequence can be found as follows:} \\
A(n) = A(1) + (n - 1)d
\]

where

\[
\begin{align*}
    n &\quad \text{is the number of the term} \\
    A(n) &\quad \text{is the } n^{th} \text{ term in the sequence} \\
    A(1) &\quad \text{is the first term in the sequence} \\
    d &\quad \text{is the common difference}
\end{align*}
\]

So given the explicit formula, \( A(n) = 3 + (n - 1)(7) \), we can identify that the starting value of the sequence is 3 and the common difference is 7.

We can find the 23\textsuperscript{rd} term in the sequence by simply evaluating the function when \( n = 23 \).

\[
\begin{align*}
    A(n) &= 3 + (n - 1)(7) \\
    A(23) &= 3 + (23 - 1)(7) \\
    A(23) &= 3 + (22)(7) \\
    A(23) &= 3 + 154 \\
    A(23) &= 157
\end{align*}
\]

So, the 23\textsuperscript{rd} term is 157.
Writing an Explicit Formula

Write the explicit formula and then use the formula to find the 15\textsuperscript{th} term for the sequence −54, −42, −30, −18, ...

Identify the first term and the common difference.

\[ \begin{align*}
-54, & \quad -42, \quad -30, \quad -18, \ldots \\
+12, & \quad +12, \quad +12
\end{align*} \]

First term: −54

Common difference: 12

Use the first term and the common difference to write the explicit formula.

\[ A(n) = A(n) + (n - 1)d \]

\[ A(n) = -54 + (n - 1)(12) \]

Use the formula to find the 15\textsuperscript{th} term.

\[ A(15) = -54 + (15 - 1)(12) \]

\[ A(15) = -54 + (14)(12) \]

\[ A(15) = -54 + 168 \]

\[ A(15) = 114 \]

The 15\textsuperscript{th} term is 114.

Determine what term −210 is in the sequence 75, 60, 45, 30, ...

1. Identify the first term and the common difference.

\[
\begin{align*}
75, & \quad 60, \quad 45, \quad 30, \ldots \\
A(1) & = 75 \\
d & = -15
\end{align*}\]

2. Write the explicit formula using the first term and common difference.

\[ A(n) = A(n) + (n - 1)d \]

\[ A(n) = 75 + (n - 1)(-15) \]

3. Substitute −210 in the explicit formula for A(n) and solve for n.

\[ A(n) = 75 + (n - 1)(-15) \]

\[ -210 = 75 + (n - 1)(-15) \]

\[ -285 = (n - 1)(-15) \]

\[ 19 = (n - 1) \]

\[ 20 = n \]

−210 is the 20\textsuperscript{th} term in the sequence.
Comparing Rates of Change: Lesson Summary with Examples

Rate of change describes the relationship between two changing quantities.

\[
\text{rate of change} = \frac{\text{change in the dependent variable}}{\text{change in the independent variable}}
\]

When comparing rates of change in tables and graphs or between intervals, make sure you are looking at the rate of change values after finding the absolute value.

A line has a constant rate of change, which is the slope.

Reviewing Rate of Change

The following table shows the average number of text messages a U.S. teenager (ages 13 to 17) makes each month. We can calculate the rate of change for an interval in this table, that is, from one row to the next. Find the rate of change from 2007 to 2008.

<table>
<thead>
<tr>
<th>Year</th>
<th>Average Number of Text Messages per Month</th>
<th>rate of change = \frac{\text{change in the dependent variable}}{\text{change in the independent variable}}</th>
</tr>
</thead>
<tbody>
<tr>
<td>2007</td>
<td>435</td>
<td>rate of change = \frac{\text{change in number of texts}}{\text{change in year}}</td>
</tr>
<tr>
<td>2008</td>
<td>1742</td>
<td>rate of change = \frac{1742 - 435}{2008 - 2007}</td>
</tr>
<tr>
<td>2009</td>
<td>2899</td>
<td>rate of change = \frac{1307}{1}</td>
</tr>
<tr>
<td>2010</td>
<td>3339</td>
<td>rate of change = 1307 texts per month</td>
</tr>
</tbody>
</table>

The first interval shows that from 2007 to 2008, the average number of text messages sent per month increased by 1307.
Constant Rate of Change

The graph of a line always has a constant rate of change. If data has a constant rate of change, the ordered pairs all lie on a line.

The rate of change of a line can be found by counting the rise and the run.

From the point (−3, 4), count down 5 units and right 6 units to get to the point (3, −1).

So the rise is −5 and the run is 6.

\[
slope = \frac{\text{rise}}{\text{run}} = \frac{-5}{6}
\]

The slope is \(\frac{-5}{6}\).

Comparing Tables and Graphs

You can compare the rate of change of tables and graphs.

![Graph comparing tables and graphs](image)
Slope-Intercept Form: Lesson Summary with Examples

To graph a line in slope-intercept form:
1. Identify the slope and the y-intercept.
2. Plot the y-intercept.
3. Plot another point by counting the slope from the y-intercept.
4. Draw the line.

A linear function can also be written in function notation. $f(x) = mx + b$ and $y = mx + b$ represent the same linear function.

Remember, when given a table of data to first identify if the data is linear by determining if there is a constant rate of change.

Parallel lines have the same slope. $m_1 = m_2$

Perpendicular lines have negative reciprocal slopes. $m_1 \cdot m_2 = -1$
To write a linear equation in slope-intercept form:

- Given the slope and \(y\)-intercept
  
  Use the values for slope and \(y\)-intercept to write the equation.

- Given the slope and a point.
  1. Substitute the slope and the point into the equation, then solve for \(b\).
  2. Substitute the slope and \(y\)-intercept into slope-intercept form.

- Given two points.
  1. Find the slope.
  2. Substitute the slope and one of the points into the equation, then solve for \(b\).
  3. Substitute the slope and \(y\)-intercept into slope-intercept form.

---

**Graphing in Slope-Intercept Form**

Graph \(y = \frac{3}{2}x - 4\) using the steps below:

1. Identify the slope and \(y\)-intercept.
2. Plot the \(y\)-intercept.
3. Plot another point by counting the slope from the \(y\)-intercept.
4. Draw the line.

![Graph of \(y = \frac{3}{2}x - 4\)](image)
Writing Linear Equations in Slope-Intercept Form

Write the equation of a line in slope-intercept form with a slope of 4 and that passes through the point (6, –3).

1. Substitute the slope, \( m = 4 \), and the coordinates from the ordered pair, (6, –3), into the equation. 
   \[ y = mx + b \]
   \[ -3 = 4(6) + b \]
   \[ b = -27 \]

2. Solve for \( b \).
   \[ b = -27 \]

3. Use the value found for the \( y \)-intercept (\( b \)) and the given slope (\( m \)) to write the equation in slope-intercept form.
   \[ y = 4x - 27 \]

Slope-Intercept Form and Two Points

Write the equation of the line in slope-intercept form that passes through the points (1, –2) and (–3, 5).

1. Find the slope.
   \[ m = \frac{5 - (-2)}{-3 - 1} \]
   \[ m = \frac{7}{4} \]

2. Substitute the slope and one of the points into the equation, then solve for \( b \).
   \[-2 = \frac{7}{4}(1) + b \]
   \[-2 = \frac{7}{4} + b \]
   \[ b = -\frac{1}{4} \]

3. Substitute the slope and \( y \)-intercept into slope-intercept form.
   \[ y = mx + b \]
   \[ y = \frac{7}{4}x - \frac{1}{4} \]
Standard Form: Lesson Summary with Examples

To write an equation of a line in standard form, first write the equation in slope-intercept form and then convert it to standard form.

To graph a line in standard form:
1. Find the x-intercept by substituting 0 for y and solving for x.
2. Find the y-intercept by substituting 0 for x and solving for y.
3. Plot the x- and y-intercepts.
4. Draw the line.
Standard Form

To write an equation of a line in standard form, first write the equation in slope-intercept form and then convert it to standard form.

Write the equation of the line in standard form with slope $\frac{5}{6}$ and $y$-intercept $5$.

$$y = mx + b$$
$$y = \frac{5}{6}x + 5$$
$$6y = 5x + 30$$

$$-5x + 6y = 30$$
**x- and y-intercepts**

The *y*-intercept of a line is the point where the line intersects the *y*-axis. All ordered pairs for *y*-intercepts are in the form \((0, y)\).

The *x*-intercept of a line is the point where the line intersects the *x*-axis. All ordered pairs for *x*-intercepts are in the form \((x, 0)\).

---

**Graphing Using Intercepts**

Graph \(2x - 3y = -6\) using the *x*- and *y*-intercepts.

\[
\begin{align*}
\text{x-intercept} & : & 2x - 3(0) &= -6 \\
& & 2x &= -6 \\
& & x &= -3 \\
& & (-3, 0) \\
\text{y-intercept} & : & 2(0) - 3y &= -6 \\
& & -3y &= -6 \\
& & y &= 2 \\
& & (0, 2)
\end{align*}
\]
Modeling Equations in Standard Form

**Online Purchases** Teresa recently purchased games and music in an online store. The online store had games on sale for $2 and MP3 albums on sale for $4. When Teresa's mother received the bill, she found a charge of $36 for the games and music. When confronted about the bill, Teresa couldn't remember how many games and MP3 albums she purchased. Write and graph an equation that models the number of games and MP3 albums Teresa purchased. Then list three possible combinations of games and MP3 albums that Teresa may have purchased.

Let \( x \) = number of games

Let \( y \) = number of MP3 albums

\[
2x + 4y = 36
\]

**x-intercept**

\[
2x + 4(0) = 36 \\
x = 18 \\
(18, 0)
\]

**y-intercept**

\[
2(0) + 4y = 36 \\
4y = 36 \\
y = 9 \\
(0, 9)
\]

### Key Point

Check the reasonableness of your linear model. You cannot buy a portion of a game or get the sales price on a portion of an MP3 album, so only points with integer coordinates are possible values for this model.
Using Graphs and Tables to Solve Systems of Equations: Lesson Summary with Examples

A system of linear equations is a set of two or more equations that have common variables.

The solution to a system of equations is the ordered pair \((x, y)\) that satisfies every equation in the system.

<table>
<thead>
<tr>
<th>Solutions of Systems of Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>One Solution</strong></td>
</tr>
<tr>
<td><img src="image1" alt="Graph" /></td>
</tr>
<tr>
<td>The slopes are different, so the lines intersect.</td>
</tr>
</tbody>
</table>

| **No Solution**                   |
| ![Graph](image2)                  |
| The slopes are the same and the \(y\)-intercepts are different, so the lines are parallel. |

| **Infinitely Many Solutions**      |
| ![Graph](image3)                  |
| The slopes and the \(y\)-intercepts are the same, so the lines are the same. |
Systems of Linear Equations and Their Solutions

Solve the system of equations.

\[ y = -2x + 8 \]
\[ y = \frac{3}{5}x - 5 \]

To solve this system of equations, first graph both lines on the same coordinate plane.

The solution is the point where the lines intersect, since this is the point that satisfies both equations. These lines intersect at (5, –2).

Verify this solution by substituting this ordered pair into both equations.

\[ y = -2x + 8 \quad y = \frac{3}{5}x - 5 \]
\[ -2 = -2(5) + 8 \quad -2 = \frac{3}{5}(5) - 5 \]
\[ -2 = -10 + 8 \quad -2 = 3 - 5 \]
\[ -2 = -2 \quad \checkmark \quad -2 = -2 \quad \checkmark \]

The solution to the system of equations is (5, –2).

Types of Solutions

A system of equations that intersects at one point has one solution. But a system of equations can also have no solution or infinitely many solutions.

\[ y = -\frac{2}{3}x + 6 \quad y = -\frac{2}{3}x + 1 \]
\[ y = \frac{1}{4}x + 2 \quad x - 4y = -8 \]
Given the equations of the lines, 3 students compared the slopes and the $y$-intercepts to determine if the system has no solution, one solution, or infinitely many solutions.

\[ -2x + 3y = -6 \]
\[ 4x - 6y = -6 \]

**Student answer:**
I can determine the number of solutions to this system by writing both equations in slope-intercept form and comparing the slopes and $y$-intercepts.

\[ -2x + 3y = -6 \quad \rightarrow \quad y = \frac{2}{3}x - 2 \]
\[ 4x - 6y = -6 \quad \rightarrow \quad y = -\frac{2}{3}x + 1 \]

Since the slopes are the same, but the $y$-intercepts are different, these lines will be parallel. I know that parallel lines never intersect, so this system has no solution.

\[ y = 4x - 3 \]
\[ 3x - y = 4 \]

**Student answer:**
The first equation is already written in slope-intercept form and I can write the second equation as $y = 3x - 4$.
The first equation has a slope of 4 while the second equation has a slope of 3. Since these slopes are different, the two lines will intersect. So the system must have one solution.

\[ y = -2(x - 2) \]
\[ 2x + y = 4 \]

**Student answer:**
By writing both equations in slope-intercept form, I can compare the slopes and $y$-intercept.

\[ y = -2(x - 2) \quad \rightarrow \quad y = -2x + 4 \]
\[ 2x + y = 4 \quad \rightarrow \quad y = -2x + 4 \]

These lines are the same, so this system has infinitely many solutions.
Testing Solutions for Correctness

When given an ordered pair, you can determine if it is a solution to the system of equations by seeing if it satisfies both equations.

Determine if $(1, -1)$ is a solution to the system of equations.

\[ y = -4x + 3 \]
\[ y = -x + 6 \]

1. Substitute $(1, -1)$ into the first equation.

\[
\begin{align*}
43 & \quad 43 \\
1 & \quad 11 \\
3 & \quad 11
\end{align*}
\]

$(1, -1)$ satisfies the first equation.

2. Substitute $(1, -1)$ into the second equation.

\[
\begin{align*}
6 & \quad 6 \\
1 & \quad 1 \\
6 & \quad 15
\end{align*}
\]

$(1, -1)$ does not satisfy the second equation.

3. Is $(1, -1)$ a solution to this system of equations?

No. $(1, -1)$ is not a solution to this system of equations since it does not satisfy both equations.
Solving Systems Using Tables

Besides graphing, we can also use tables to solve a system of equations. Solve the system of equations using tables.

\[ y = x + 2 \]
\[ y = -\frac{1}{2} x + 5 \]

Complete a table of values for each equation.

\[
\begin{array}{c|c}
 x & y \\
-2 & 0 \\
-1 & 1 \\
0 & 2 \\
1 & 3 \\
2 & 4 \\
\end{array}
\quad
\begin{array}{c|c}
 x & y \\
-2 & 6 \\
0 & 5 \\
2 & 4 \\
4 & 3 \\
6 & 2 \\
\end{array}
\]

Comparing the tables, the ordered pair (2, 4) appears in both tables. So (2, 4) is the solution to this system of equations.
Using Substitution to Solve Systems of Equations: Lesson Summary with Examples

**Solving a System of Equations by Substitution**

- **Step 1:** Isolate a variable in one of the equations, if necessary.
- **Step 2:** Replace the variable in the second equation by substituting the expression from Step 1. Then solve the new equation.
- **Step 3:** Solve for the other variable in either of the original equations.
- **Check:** After you find the solution, you can always check your ordered pair in the other original equation.

**Solving by Substitution**

- If the result is a false statement, then the system has **no solution**.
- If the result is a true statement, then the system has **infinitely many solutions**.
The Substitution Method

Solve by substitution.

Equation 1: \( y = -x + 5 \)
Equation 2: \( y = 3x - 2 \)

Step 1: The variable \( y \) is already isolated in both equations. So Step 1 is complete.

Step 2: Replace the variable in Equation 2 by substituting the expression from Step 1 and solve the equation.
\[
-x + 5 = 3x - 2 \\
5 = 4x - 2 \\
7 = 4x \\
x = \frac{7}{4}
\]

Step 3: Solve for the remaining variable in either Equation 1 or 2.
\[
y = -x + 5 \\
y = -\left(\frac{7}{4}\right) + 5 \\
y = -\frac{7}{4} + \frac{20}{4} \\
y = \frac{13}{4}
\]

Solution: \( \left(\frac{7}{4}, \frac{13}{4}\right) \)
No Solution and Infinitely Many Solutions

When graphing a system of equations, we saw that no solution meant the lines were parallel, while infinitely many solutions meant that the lines were the same.

No Solution

\[ y = -x + 3 \]
\[ 2x + 2y = -2 \]

Infinitely Many Solutions

\[ y = \frac{2}{3}x + 3 \]
\[ 2x - 3y = -9 \]

Solve each system of equations by substitution.

\[ y = -x + 3 \]
\[ 2x + 2y = -2 \]
\[ 2x + 2(-x + 3) = -2 \]
\[ 2x - 2x + 6 = -2 \]
\[ 6 = -2 \text{ false} \]

no solution

\[ y = \frac{2}{3}x + 3 \]
\[ 2x - 3y = -9 \]
\[ 2x - 3 \left( \frac{2}{3}x + 3 \right) = -9 \]
\[ 2x - 2x - 9 = -9 \]
\[ -9 = -9 \text{ true} \]

infinitely many solutions
Using Elimination to Solve Systems of Equations: Lesson Summary with Examples

To solve a system of equations using elimination:

1. If necessary, multiply one or both equations so that the coefficients of one of the variables will be eliminated when adding or subtracting.
2. Add or subtract the equations to eliminate one of the variables.
3. Solve for the remaining variable in the resulting equation.
4. Substitute that solution into one of the original equations and solve for the other variable.
5. Write the solution as an ordered pair.

Key Point
To determine what number to multiply each equation by, identify the least common multiple (LCM) of the coefficients of the variable you want to eliminate.

Elimination Using Addition
To solve a system of equations using elimination, add or subtract the equations to eliminate one variable. Then solve for the other.

Solve this system of equations using elimination.

\[4x - 3y = 7\]
\[-4x - 5y = 9\]

Add the equations to eliminate the variable \(x\).

\[
\begin{align*}
4x - 3y &= 7 \\
(+)(-4x - 5y &= 9) \\
0 - 8y &= 16
\end{align*}
\]

Substitute \(-2\) for \(y\) into either equation.

\[
\begin{align*}
4x - 3y &= 7 \\
4x - 3(-2) &= 7 \\
4x &= 14
\end{align*}
\]

Check the ordered pair in the other equation.

\[
\begin{align*}
-4x - 5y &= 9 \\
-4\left(-\frac{1}{4}\right) - 5(-2) &= 9 \\
1 + 10 &= 9
\end{align*}
\]

Solution: \(\left(\frac{1}{4}, -2\right)\)
Elimination Using Subtraction

In this system of equations, adding the equations will not eliminate either variable.

\[ 2x - 7y = 29 \]
\[ 2x + y = 5 \]

The coefficient of \( x \) is the same in both equations, so adding the equations will give us \( 4x \). This won’t eliminate the variable. But subtracting the equations will.

Subtract the equations. Rewrite as an addition problem by distributing the negative sign.

\[
\begin{align*}
2x - 7y &= 29 \\
(-) 2x + y &= 5 \\
\hline
0 - 8y &= 24
\end{align*}
\]

Substitute \(-3\) for \( y \) into either equation.

\[
\begin{align*}
2x &= 8 \\
y &= -3
\end{align*}
\]

Solution: \((4, -3)\)

Note: You do not have to rewrite the system using addition, but you may find it helpful so that you do not make an error when subtracting.

Elimination with Multiplication of One Equation

In this case, we cannot eliminate \( \textbf{either} \) of the variables simply by adding or subtracting. But we can still solve this system using elimination after multiplying one of the equations.

\[
\begin{align*}
4c + 8d &= -16 \\
5c - 2d &= -26
\end{align*}
\]

To eliminate the variable \( d \), multiply the second equation by 4. Substitute \(-5\) for \( c \) into either equation.

\[
\begin{align*}
4c + 8d &= -16 \\
5c - 2d &= -26 \quad \text{multiply by 4} \\
(+) 20c - 8d &= -104 \\
24c &= -120 \\
c &= -5 \\
5(-5) - 2d &= -26 \\
-25 - 2d &= -26 \\
-2d &= -1 \\
d &= \frac{1}{2}
\end{align*}
\]

Solution: \((-5, \frac{1}{2})\)

Note: You could also multiply by \(-4\), but it means that you will need to subtract the equations, rather than add, in order to eliminate the variable \( y \). But you will get the same solution either way.
Elimination with Multiplication of Both Equations

There are systems of equations that require you to multiply both equations by a constant in order to solve using elimination. Identify the least common multiple for the variable you want to eliminate, then multiply each equation.

\[3x - 2y = 31\]
\[2x + 5y = 8\]

To eliminate the variable \(x\), the LCM of the coefficients of \(x\) is 6. So multiply the first equation by 2 and the second equation by −3.

\[
\begin{align*}
3x - 2y &= 31 \quad \text{multiply by 2} \\
2x + 5y &= 8 \quad \text{multiply by -3}
\end{align*}
\]

\[
\begin{align*}
6x - 4y &= 62 \\
(-6x - 15y &= -24) \\
-19y &= 38 \\
y &= -2
\end{align*}
\]

Substitute −2 for \(y\) into either equation.

\[
\begin{align*}
2x + 5(-2) &= 8 \\
2x - 10 &= 8 \\
x &= 9
\end{align*}
\]

The solution is \((9, -2)\).
Applications of Systems of Equations: Lesson Summary with Examples

Different problems may be easier to solve using one method versus another.

This chart can help you determine which method to try when solving a system of equations.

The Four Methods for Solving a System of Equations

There are four different ways of solving a system of equations.

1. Tables
2. Graphing
3. Substitution
4. Elimination
Applying Systems of Equations

Systems of equations can be used to model various real-world situations. First, write the equations that represent the system, and then decide the best method to solve the system. After finding the solution to the system, interpret the solution within the context of the problem.

Basketball Tyrone scored 83 points in last week’s basketball tournament, a combination of 2-point and 3-point shots. The number of 2-point shots he scored was twice his 3-point shots plus 3. How many 2-point and 3-point shots did Tyrone make during the tournament?

Let $x$ equal the number of 2-point shots made and let $y$ equal the number of 3-point shots made.

First, write an equation for the total points Tyrone scored.

$$2x + 3y = 83$$

Now, write an equation that relates the number of 2-point shots made to the number of 3-point shots made.

$$x = 2y + 3$$

Solve the system of equations using substitution.

$$2(2y + 3) + 3y = 83$$

$$4y + 6 + 3y = 83$$

$$7y = 77$$

$$y = 11$$

$$x = 22 + 3$$

$$x = 25$$

Solution: $(25, 11)$

Tyrone scored 25 2-point shots and 11 3-point shots.

Distance-Rate Problems

Systems of equations can also be used to solve problems involving distance. The formula, $d = rt$, shows that the distance traveled $d$ is equal to the product of the rate $r$ and time $t$.

Travel Bob and Janet are traveling to spend the weekend at their daughter’s house in another state. They are both leaving from home, but have to drive separately. Bob averages 75 mph, but starts an hour after Janet. Janet averages 60 mph. How many miles will they travel before they meet each other? How much time will Janet have been driving when they meet?

Write a system of equations in which $d$ represents the distance traveled by each person and $t$ represents the number of hours after Janet left.

Janet averages 60 mph, so we can write the equation, $d = 60t$, to represent Janet’s distance $d$ after time $t$.

Bob averages 75 mph, but starts an hour after Janet. Since he leaves an hour after Janet, his time traveled is $t – 1$. So we can write the equation $d = 75(t – 1)$ to represent Bob's distance $d$.

Now, solve the system of equations using substitution.

$$75(t – 1) = 60t$$

$$75t – 75 = 60t$$

$$–75 = –15t$$

$$t = 5$$

Solution: $(300, 5)$

They will travel 300 miles when they meet each other. Janet will have been traveling 5 hours when they meet.
This problem could also have been solved using a table. Just be careful when writing Bob's distance as he leaves an hour after Janet.

<table>
<thead>
<tr>
<th>Number of hours after Janet left</th>
<th>Janet's distance</th>
<th>Bob's distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>60</td>
<td>--</td>
</tr>
<tr>
<td>2</td>
<td>120</td>
<td>75</td>
</tr>
<tr>
<td>3</td>
<td>180</td>
<td>150</td>
</tr>
<tr>
<td>4</td>
<td>240</td>
<td>225</td>
</tr>
<tr>
<td>5</td>
<td>300</td>
<td>300</td>
</tr>
</tbody>
</table>

In the previous problem, Bob and Janet were traveling in the same direction. If they travel in opposite directions, use the formula \( d = rt \), but think carefully about how to define the variables and set up the equations.

**Travel** Two cars leave the same location at the same time traveling in opposite directions. One car is traveling 14 mph faster than the other. In 5 hours, they are 650 miles apart. Write the system of equations to find the rate of each car.

<table>
<thead>
<tr>
<th></th>
<th>Distance</th>
<th>Rate</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slower car</td>
<td>( d )</td>
<td>( r )</td>
<td>5</td>
</tr>
<tr>
<td>Faster car</td>
<td>( 650 - d )</td>
<td>( r + 14 )</td>
<td>5</td>
</tr>
<tr>
<td>Total</td>
<td>650</td>
<td>--</td>
<td>5</td>
</tr>
</tbody>
</table>

Use the formula \( d = rt \)

Equation for the slower car: \( d = r(5) \)
Equation for the faster car: \( 650 - d = (r + 14)(5) \)
The jet stream is a current of wind that typically moves across the United States from west to east. So when a plane flies from the West Coast to the East Coast, this jet stream creates a tailwind that increases the plane’s ground speed. But if the plane flies from east to west, this jet stream creates a headwind that decreases the plane’s ground speed.

**Tailwind:**
A plane flying with the jet stream has a tailwind. The jet stream actually makes the plane go faster. So the ground speed of the plane equals its air speed plus the wind speed.

**Headwind:**
A plane flying against the jet stream has a headwind that actually slows down the plane’s speed. So the ground speed equals the air speed of the plane minus the wind speed.

![Tailwind Diagram](image)

![Headwind Diagram](image)

**The Break-Even Point**
The break-even point for a business is the point at which the operating costs are equal to the revenue. When costs are higher than revenue, a company has a loss. If revenue is higher than cost, the company has a profit.

![Break-Even Graph](image)

Use a system of equations to determine the break-even point for a business.
Better Value Problems

Systems of equations can often be used to represent situations that require a person to figure out which option is the better buy.

**Skateboarding** A new skateboard park, StreetStyle, just opened up near Evan's house. It offers an annual pass for $300 that includes an unlimited number of skate sessions. Evan could also purchase individual sessions for $4 each after a one-time annual membership fee of $30.

Let $x$ equal the number of individual sessions and $y$ equal the total cost to skate per year. Write a system of equations that models the two different payment options.

**Annual Pass**
Verbal Model: Total cost = cost of the annual pass
Equation: $y = 300$

**Individual Sessions**
Verbal Model: Total cost = (cost per session)(number of sessions) + membership fee
Equation: $y = 4x + 30$

Solve the system of equations using substitution.

\[
\begin{align*}
300 &= 4x + 30 \\
270 &= 4x \\
67.5 &= x
\end{align*}
\]

Evan would have to go to StreetStyle 68 times to make the annual pass worthwhile.

*Note:* Evan cannot attend half a session, so he must go 68 times for the annual pass to be a better deal than individual sessions.
Linear Inequalities and Systems of Linear Inequalities: Lesson Summary with Examples

Linear Inequalities

A linear inequality in two variables can be written in the form $y < mx + b$, $y > mx + b$, $y \leq mx + b$, or $y \geq mx + b$. The solution of a linear inequality is the set of ordered pairs that satisfy the inequality.

The boundary is the linear equation that separates the coordinate plane into two half-planes. The boundary may be a solid or dashed line, depending on the inequality symbol.

Systems of Linear Inequalities

A system of linear inequalities is a set of two or more linear inequalities with the same variables. The solution of a system of linear inequalities is the set of ordered pairs that satisfies each of the inequalities in the system.

When graphing a system of inequalities, each linear inequality is graphed on the same graph. The solution to the system is represented by the intersection of the solutions of each linear inequality.

- **Step 1**: Graph the first linear inequality.
- **Step 2**: Graph the second linear inequality.
- **Step 3**: Shade the intersection of the two shaded regions.

Steps for Graphing a Linear Inequality

1. **Step 1**: Graph the boundary using the related linear equation.
   - If the inequality symbol is $\leq$ or $\geq$, use a solid boundary line.
   - If the inequality symbol is $<$ or $>$, use a dashed boundary line.

2. **Step 2**: Test a point not on the boundary to determine which half-plane to shade.

3. **Step 3**: Shade the appropriate region.
Graphing Inequalities with Two Variables

Graph \( y < 2x - 1 \).

Step 1: Graph the related equation \( y = 2x - 1 \).  
The boundary line is dashed since the inequality symbol is \textit{less than}.

Step 2: Test a point not on the boundary line to determine which half-plane to shade.  
sample test point \((0, 0)\)  
\[ y < 2x - 1 \]  
\[ ? \]  
\[ 0 < 2(0) - 1 \]  
\[ ? \]  
\[ 0 < 0 - 1 \]  
\[ 0 < -1 \] false

Step 3: Shade the appropriate region.

While using a test point will always help you determine which half-plane to shade, you can also use the direction of the inequality symbol to determine which side of the boundary to shade.

Solve the inequality for \( y \):

- If the symbol is \(<\) or \(\leq\), shade \textit{below} the line.
- If the symbol is \(>\) or \(\geq\), shade \textit{above} the line.

Graphing Linear Inequalities in One Variable

Graphing a linear inequality in one variable is no different than graphing a linear inequality in two variables. 
The boundary will be either a horizontal or vertical line.
Applications of Linear Inequalities

Below are a few common phrases that represent inequality symbols.

<table>
<thead>
<tr>
<th></th>
<th>&lt;</th>
<th>&gt;</th>
<th>≤</th>
<th>≥</th>
</tr>
</thead>
<tbody>
<tr>
<td>less than</td>
<td>greater than</td>
<td>less than or equal to</td>
<td>greater than or equal to</td>
<td></td>
</tr>
<tr>
<td>fewer than</td>
<td>more than</td>
<td>at most</td>
<td>at least</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>no more than</td>
<td>no less than</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>no greater than</td>
<td>no fewer than</td>
<td></td>
</tr>
</tbody>
</table>

**Landscaping** Daniel makes some extra money by doing small landscaping jobs for people in his neighborhood. He buys mulch and fertilizer from the local landscaping store. He pays $3 per pound for mulch and $1 per pound for premium fertilizer. If one customer wants to spend less than $20 on these supplies, how many pounds of each can he buy?

Let \( x \) = the number of pounds of mulch and let \( y \) = the number of pounds of fertilizer.

First, write an inequality that models the situation. \( 3x + y < 20 \).

Now graph the inequality.

Think about what this graph tells us. If the ordered pair (1, 2) is a solution on the graph, it means that Daniel could buy 1 pound of mulch and 2 pounds of fertilizer.
Systems of Linear Inequalities

Graph the system of linear inequalities.

\[ y < 4x - 6 \]
\[ y \geq -3x + 1 \]

Graph each linear inequality. Then shade the intersection of the two shaded regions.

Applications of Systems of Inequalities

Gifts Naomi is buying holiday gifts for her friends. Her favorite store is selling earrings for $8 each and scarves for $10 each. She needs to buy at least 7 gifts, but wants to spend no more than $100.

First, write a system of linear inequalities that model this situation. Let \( x \) equal the number of pairs of earrings and \( y \) equal the number of scarves that Naomi buys.

Number of gifts: \( x + y \geq 7 \)
Total spent: \( 8x + 10y \leq 100 \)

Now, graph the system of linear inequalities.

Some of the possible combinations of earrings and scarves that Naomi can buy are 0 pairs of earrings and 10 scarves or 5 pairs of earrings and 5 scarves.
Linear Programming: Lesson Summary with Examples

Linear programming uses linear inequalities, or constraints, to find the maximum and minimum values for a particular situation.

The feasible region is the area of intersection of the linear inequalities when all of the constraints are met. The objective function is a model of the quantity that you want to maximize or minimize.

Steps for finding the maximum and minimum values of an objective function given a system of inequalities

Step 1: Graph the constraints, or system of linear inequalities.
Step 2: Identify the vertices of the feasible region.
Step 3: Evaluate the objective function using the vertices.
Step 4: Identify the maximum and minimum values.

Linear Programming

Linear programming is a method for finding the maximum and minimum values of some quantity. Once the constraints are graphed, the feasible region can then be identified. The feasible region can be bounded or unbounded. The vertices of the feasible region are the points at which one boundary line of the system of linear inequalities intersects another boundary line of the system.

The bounded feasible region has three vertices: (0, 3), (4, 1), and (4, 5).

The unbounded feasible region has two vertices: (1, –3) and (1, 3).
**Finding the Maximum and Minimum Values for Bounded Regions**

Evaluate the objective function using each vertex of the feasible region to find the maximum and minimum values for the system of linear inequalities.

- **Vertex (−3, 4):**
  \[ f(x, y) = 2x + 3 \]
  \[ f(-3, 4) = 2(-3) + 3(4) = 6 \]
  \[ f(-3, 4) = 6 + 12 = 18 \]
- **Vertex (3, 4):**
  \[ f(x, y) = 2x + 3 \]
  \[ f(3, 4) = 2(3) + 3(4) = 6 + 12 = 18 \]

- **Vertex (−5, −4):**
  \[ f(x, y) = 2x + 3 \]
  \[ f(-5, -4) = 2(-5) + 3(-4) = -10 - 12 = -22 \]
- **Vertex (5, −4):**
  \[ f(x, y) = 2x + 3 \]
  \[ f(5, -4) = 2(5) + 3(-4) = 10 - 12 = -2 \]

The maximum value is 18 and occurs at (3, 4). The minimum value is −22 and occurs at (−5, −4).

**Finding the Maximum and Minimum Values for Unbounded Regions**

When a system of linear inequalities has an unbounded feasible region, the objective function will either have a maximum or minimum value, but not both.

Find the maximum and minimum values, if applicable.

\[ f(x, y) = 6x + 2y \]
\[ y \leq 2x + 12 \]
\[ y \geq 2x - 2 \]
\[ y \geq \frac{1}{3}x - 2 \]

- **Vertex: (−6, 0)**
  \[ f(x, y) = 6x + 2y \]
  \[ f(-6, 0) = 6(-6) + 2(0) = -36 + 0 = -36 \]
  \[ f(-6, 0) = -36 \]
  value < −36
  no minimum

- **Vertex: (0, −2)**
  \[ f(x, y) = 6x + 2y \]
  \[ f(0, -2) = 6(0) + 2(-2) = 0 - 4 = -4 \]
  \[ f(0, -2) = -4 \]
  value > −4
  no maximum

- **Test Point: (0, 0)**
  \[ f(x, y) = 6x + 2y \]
  \[ f(0, 0) = 6(0) + 2(0) = 0 \]
  \[ f(0, 0) = 0 \]
  The minimum value is −36.
Graphing a Systems of Linear Inequalities to Find the Maximum and Minimum Values

Find the maximum and minimum values for the system of linear inequalities with the given objective function.

\[ f(x,y) = 8x - 2y \]
\[ x + y \geq 2 \]
\[ y \geq -3 \]
\[ x \leq 3 \]
\[ x \geq -1 \]

Step 1: Graph the constraints, or system of linear inequalities.
\[ x + y \geq 2 \]
\[ y \geq -3 \]
\[ x \leq 3 \]
\[ x \geq -1 \]

Step 2: Identify the vertices of the feasible region.

The vertices of the feasible region are (–1, –3), (3, –1), (–1, 3), and 3, –3).

Step 3: Evaluate the objective function using the vertices.

<table>
<thead>
<tr>
<th>Coordinates of vertices of feasible region</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(–1, –3)</td>
<td>–2</td>
</tr>
<tr>
<td>(3, –1)</td>
<td>26</td>
</tr>
<tr>
<td>(–1, 3)</td>
<td>–14</td>
</tr>
<tr>
<td>(3, –3)</td>
<td>30</td>
</tr>
</tbody>
</table>

Step 4: Identify the maximum and minimum values.

The maximum value is 30.
The minimum value is –14.
Solving Real-World Situations Using Linear Programming

Christian is raising money for charity by raking leaves and mowing lawns for his neighbors. He asks for donations of $7 for raking leaves and $5 for mowing lawns. Each week he mows and rakes for no more than 15 lawns. Christian will mow at least 1 lawn and rake between 1 and 6 lawns. How can Christian maximize his weekly earnings for charity?

First, write a system of linear inequalities that model this situation. Let \( m \) equal the number of lawns he can mow and \( r \) equal the number of lawns he can rake.

- Each week he mows and rakes for no more than 15 lawns: \( m + r \leq 15 \)
- Christian will mow at least 1 lawn: \( m \geq 1 \)
- He usually rakes between 1 and 6 lawns: \( 1 \leq r \leq 6 \)

Next, write the objective function to be maximized.

Since Christian is trying to maximize his weekly earnings for charity by mowing and raking, the objective function consists of the money he will earn from mowing and raking.

Next, write the objective function to be maximized.

\( f(r, m) = 7r + 5m \)

Now, graph the constraints and identify the feasible region.

Then, identify the vertices of the feasible region and evaluate the objective function.

<table>
<thead>
<tr>
<th>coordinates of vertices of feasible region</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>12</td>
</tr>
<tr>
<td>(1, 14)</td>
<td>77</td>
</tr>
<tr>
<td>(6, 9)</td>
<td>87</td>
</tr>
<tr>
<td>(6, 1)</td>
<td>47</td>
</tr>
</tbody>
</table>

Finally, identify the maximum value and answer the question.

The maximum value of 87 occurs at (6, 9). That means, Christian can earn as much as $87 by raking 6 yards and mowing 9 lawns.
### Properties of Exponents: Lesson Summary with Examples

**Summary of Properties of Exponents**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Zero Exponent Property</strong></td>
<td>For any nonzero number $a$, $a^0 = 1$</td>
</tr>
<tr>
<td><strong>Negative Exponent Property</strong></td>
<td>For all numbers $a$ and $n$ where $a \neq 0$, $a^{-n} = \frac{1}{a^n}$</td>
</tr>
<tr>
<td><strong>Product of Powers Property</strong></td>
<td>For all numbers $a$, $m$, and $n$ where $a \neq 0$, $a^m \cdot a^n = a^{m+n}$</td>
</tr>
<tr>
<td><strong>Power of a Power Property</strong></td>
<td>For all numbers $a$, $m$, and $n$ where $a \neq 0$, $(a^m)^n = a^{mn}$</td>
</tr>
<tr>
<td><strong>Power of a Product Property</strong></td>
<td>For all numbers $a$, $b$, and $m$ where $a \neq 0$ and $b \neq 0$, $(ab)^m = a^m b^m$</td>
</tr>
<tr>
<td><strong>Quotient of Powers Property</strong></td>
<td>For all numbers $a$, $m$, and $n$ where $a \neq 0$, $a^m \div a^n = a^{m-n}$</td>
</tr>
<tr>
<td><strong>Power of a Quotient Property</strong></td>
<td>For all numbers $a$, $b$, and $m$ where $a \neq 0$ and $b \neq 0$, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$</td>
</tr>
<tr>
<td><strong>Exponent Property of Equality</strong></td>
<td>For all numbers $b$, $x$, and $y$ where $b \neq 0$: If $b^x = b^y$, then $x = y$.</td>
</tr>
</tbody>
</table>

**Key Point**

For all numbers $a$, $b$, and $m$ where $a \neq 0$ and $b \neq 0$, $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$
Exponents

Write each expression using exponents.

\[ 3 \cdot 3 \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y = 3^2x^3y^4 \]
\[ \frac{7 \cdot 7 \cdot 7 \cdot a \cdot a \cdot a \cdot c \cdot c}{b \cdot b \cdot b} = \frac{7^4a^3c^2}{b^3} \]

Zero and Negative Exponents

**Zero Exponent Property**

For any nonzero number \( a \),

\[ a^0 = 1 \]

Simplify each expression using the Zero Exponent Property.

\[(4.7)^0 = 1 \quad \left(\frac{2}{3}\right)^0 = 1 \quad (-6)^0 = 1 \quad 3ab^0 = 3a \quad (3ab)^0 = 1\]

**Negative Exponent Property**

For all numbers \( a \) and \( n \) where \( a \neq 0 \),

\[ a^{-n} = \frac{1}{a^n} \]

Using the Negative Exponent Property, write an equivalent expression for \( 2^{-7} \).

\[ 2^{-7} = \frac{1}{2^7} \quad 2^{-7} = \frac{1}{128} \]
Product of Powers Property

<table>
<thead>
<tr>
<th>Product of Powers Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>For all numbers (a, m,) and (n) where (a \neq 0,) (\Rightarrow a^m \cdot a^n = a^{m+n})</td>
</tr>
</tbody>
</table>

Simplify \((8u^7v^2)(-2u^3)(5v^{-4})\).

1. Multiply the coefficients.
   \[
   (8u^7v^2)(-2u^3)(5v^{-4})
   \]
   \[
   (8)(-2)(5) = -80
   \]

2. Add the exponents of the powers with base \(u\).
   \[
   u^7 \cdot u^3 = u^{7+3}
   \]
   \[
   -80u^{10}
   \]

3. Add the exponents of the powers with base \(v\).
   \[
   v^2 \cdot v^{-4} = v^{2+-4}
   \]
   \[
   -80u^{10}v^2
   \]

4. Write the expression using positive exponents.
   \[
   -80u^{10}v^{-2}
   \]
   \[
   \frac{-80u^{10}}{v^2}
   \]

Power of a Power and Power of a Product

<table>
<thead>
<tr>
<th>Power of a Power Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>For all numbers (a, m,) and (n) where (a \neq 0,) (\Rightarrow (a^m)^n = a^{mn})</td>
</tr>
</tbody>
</table>

Simplify an expression using the Power of a Power Property.

Simplify the expression. \((x^3)^2\)
Multiply the exponents. \(x^{3 \cdot 2}\)
Simplify. \(x^6\)
Simplify an expression using the Power of a Product Property.

Simplify the expression. \((x^5y^4)^3\)

Use the Power of a Product Property. \((x^5)^3 (y^4)^3\)

Use the Power of a Power Property. \((x^{5\cdot 3}) (y^{4\cdot 3})\)

Simplify. \(x^{15}y^{12}\)

**Quotient of Powers and Power of a Quotient**

Simplify an expression using the Quotient of Powers Property.

Simplify the expression. \(\frac{b^6}{b^8}\)

Use the Quotient Powers Property. \(b^{6-8}\)

Simplify. \(b^{-2}\)

Use the Negative Exponent Property. \(\frac{1}{b^2}\)
Simplify each expression using the Power of a Quotient Property.

\[
\left( \frac{3m^6}{5} \right)^2
\]

Simplify the expression.

\[
3^2 \left( \frac{m^6}{5} \right)^2
\]

Square each factor.

\[
\frac{9m^{12}}{25}
\]

Simplify.

\[
\left( \frac{7}{2} \right)^{-3}
\]

Simplify the expression.

\[
\frac{7^{-3}}{2^{-3}}
\]

Raise each factor to the \(-3\) power.

\[
\frac{2^3}{7^3}
\]

Write the expression using positive exponents.

\[
\frac{8}{343}
\]

Simplify.

Solving Exponential Equations

\[
\text{Exponent Property of Equality}
\]

For all numbers \(b\), \(x\), and \(y\) where \(b \neq 0\):

If \(b^x = b^y\), then \(x = y\).

Solve the equation for \(x\).

\[2^{3x-4} = 2^x\]

Set the exponents equal to one another.

\[3x - 4 = x\]

Solve for \(x\).

\[-4 = -2x\]

\[2 = x\]

Solve the equation for \(x\).

\[6^x = 36\]

Write 36 with a base of 6.

\[6^x = 6^2\]

Set the exponents equal to one another.

\[x = 2\]
Geometric Sequences: Lesson Summary with Examples

A geometric sequence is a sequence of numbers in which each term is found by multiplying the previous term by a constant. The constant value in a geometric sequence is the common ratio.

Recursive formulas are used to determine the NEXT term of a sequence from one or more of the preceding terms.

Explicit formulas are used to determine ANY term in a sequence.

Formulas for Geometric Sequences

Recursive Formula for a Geometric Sequence

Given a starting term and a common ratio, \( r \), each term in a geometric sequence can be found as follows:

\[
\text{NEXT} = \text{NOW} \cdot r
\]

Recursive Formula in Function Notation for a Geometric Sequence

Each term in a geometric sequence can be found as follows:

\[
f(n + 1) = f(n) \cdot r
\]

where

- \( n \) is the number of the term
- \( f(n) \) is the \( n^{th} \) term in the sequence
- \( f(n + 1) \) is the \( (n + 1) \) term in the sequence
- \( r \) is the common ratio
- \( f(1) \) is the starting term

Explicit Formula for a Geometric Sequence

Any term in a geometric sequence can be found as follows:

\[
G(n) = G(1) \cdot r^{n-1}
\]

where

- \( n \) is the number of the term
- \( G(n) \) is the \( n^{th} \) term in the sequence
- \( G(1) \) is the first term in the sequence
- \( r \) is the common ratio
Geometric Sequences

A geometric sequence is a sequence of numbers where the order of the numbers is determined by multiplying by a constant value called the common ratio.

\[3, \quad 9, \quad 27, \quad 81, \quad 243, \ldots\]

\[\times 3 \quad \times 3 \quad \times 3 \quad \times 3\]

Recursive Formulas

1. The first term is the starting value.
   - NEXT = NOW \cdot 2, starting at 4
   - \(1^{st}\) term

2. In the formula, substitute the first term for NOW and find the second term.
   - NEXT = 4 \cdot 2
   - NEXT = 8
   - \(2^{nd}\) term

3. Substitute the second term for NOW to find the third term.
   - NEXT = 8 \cdot 2
   - NEXT = 16
   - \(3^{rd}\) term

4. Substitute the third term for NOW to find the fourth term.
   - NEXT = 16 \cdot 2
   - NEXT = 32
   - \(4^{th}\) term

5. List all terms.

The first four terms for the geometric sequence are 4, 8, 16, and 32.

Recursive Formulas in Function Notation

Write the first four terms of the geometric sequence.

\[f(n + 1) = f(n) \cdot 2, \quad f(1) = 9\]

\[
\begin{align*}
  f(1) &= 9 \\
  f(2) &= f(1) \cdot 2 \\
  f(3) &= f(2) \cdot 2 \\
  f(4) &= f(3) \cdot 2 \\
  f(2) &= 18 \\
  f(3) &= 36 \\
  f(4) &= 72
\end{align*}
\]

The first four terms are 9, 18, 36, and 72.
Using an Explicit Formula

Find the 7th term.

Explicit formula. \( G(n) = 4 \cdot (3)^{n-1} \)
Substitute \( n = 7 \). \( G(7) = 4 \cdot (3)^{7-1} \)
Simplify. \( G(7) = 4 \cdot (3)^6 \)
\( G(7) = 4 \cdot 729 \)
\( G(7) = 2916 \)

The 7th term is 2916.

Writing an Explicit Formula

Write an explicit formula for the nth term in the geometric sequence. Then use the formula to find the 8th term.

6, −18, 54, −162, ...

1. Identify the first term and the common ratio.

   6, −18, 54, −162, ...
   first term: \( G(1) = 6 \)  
   common ratio: \( r = -3 \)

2. Write the explicit formula using the first term and common ratio.

   \( G(n) = G(1) \cdot r^{n-1} \)
   \( G(n) = (6) \cdot (-3)^{n-1} \)

3. Evaluate the function \( G(n) \) for the given term number \( n \).

   \( n = 8 \)
   \( G(n) = (6) \cdot (-3)^{n-1} \)
   \( G(8) = (6) \cdot (-3)^{8-1} \)
   \( G(8) = (6) \cdot (-3)^7 \)
   \( G(8) = (6) \cdot (-2187) \)

   The 8th term in the sequence is −13122.
Graphing Exponential Functions: *Lesson Summary with Examples*

### Exponential Function

$$y = ab^x$$

where $a \neq 0$, $b > 0$, and $b \neq 1$

Remember:
- An exponential function in the form $y = ab^x$ represents exponential *growth* when $b > 1$.
- An exponential function in the form $y = ab^x$ represents exponential *decay* when $0 < b < 1$.
- The key features for an exponential function include the domain, the range, the asymptote, and the $y$-intercept.

<table>
<thead>
<tr>
<th>Exponential Growth</th>
<th>Exponential Decay</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 3(1.5)^x$</td>
<td>$y = 2(0.5)^x$</td>
</tr>
</tbody>
</table>

- **Domain:** all real numbers
- **Range:** $y > 0$
- **Asymptote:** $y = 0$
- **$y$-intercept:** (0, 3)

- **Domain:** all real numbers
- **Range:** $y > 0$
- **Asymptote:** $y = 0$
- **$y$-intercept:** (0, 2)
• Adding the constant \( k \) to an exponential function shifts the parent function up and down the \( y \)-axis.

\[
\begin{align*}
g(x) &= 2^x \\
h(x) &= 2^x + 2
\end{align*}
\]

\[
\begin{align*}
g(x) &= 2^x \\
h(x) &= 2^x - 2
\end{align*}
\]

**Summary of Key Features**

| \( y = ab^x \) when \( a > 0 \) | • The domain is all real numbers.  
• The range is \( y > 0 \).  
• The asymptote is \( y = 0 \).  
• The \( y \)-intercept is \( (0, a) \). |
|---|---|
| \( y = ab^x \) when \( a < 0 \) | • The domain is all real numbers.  
• The range is \( y < 0 \).  
• The asymptote is \( y = 0 \).  
• The \( y \)-intercept is \( (0, a) \). |
| \( y = b^x + k \) | • The domain is all real numbers.  
• The range is \( y > k \).  
• The asymptote is \( y = k \).  
• The \( y \)-intercept is \( (0, 1 + k) \). |
Graphing Exponential Functions

To graph an exponential function

1. Generate a table of values.
2. Plot the ordered pairs on the coordinate plane.
3. Sketch the curve.

Key features:

- The domain is all real numbers.
- The range is \( y > 0 \).
- The asymptote is \( y = 0 \).
- The \( y \)-intercept is \( (0, 1) \).
Graph $y = \left(\frac{1}{2}\right)^x$.

Key features:
- The domain is all real numbers.
- The range is $y > 0$.
- The asymptote is $y = 0$.
- The $y$-intercept is $(0, 1)$.

Graphing $y = b^x + k$

To graph an exponential function using a vertical shift:
1. Graph the parent function.
2. Shift each point of the curve down 4 units.
3. Draw the curve.

Graph $y = 3^x - 4$. 
Applications of Exponential Functions: *Lesson Summary with Examples*

The growth factor is 1 plus the percent rate of change, written as a decimal. The decay factor is 1 minus the percent rate of change, written as a decimal.

<table>
<thead>
<tr>
<th>Growth Factor</th>
<th>Decay Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = 1 + r$</td>
<td>$b = 1 - r$</td>
</tr>
</tbody>
</table>

### Exponential Growth

\[
y = ab^x \\
y = a(1 + r)^x
\]

- $y$ is the final amount
- $a$ is the initial amount
- $b$ is the growth factor
- $x$ is the time
- $r$ is the percent rate of change, written as a decimal

### Exponential Decay

\[
y = ab^x \\
y = a(1 - r)^x
\]

- $y$ is the final amount
- $a$ is the initial amount
- $b$ is the decay factor
- $x$ is the time
- $r$ is the percent rate of change, written as a decimal

### Compound Interest Formula

\[
A = P\left(1 + \frac{r}{n}\right)^{nt}
\]

- $A$ is the final balance
- $P$ is the principal
- $r$ is the rate, written as a decimal
- $n$ is the number of times interest is compounded each year
- $t$ is the time, in years

### Exponential Function for Half-life

\[
y = a(0.5)^{\frac{t}{x}}
\]

- $y$ is the mass in grams remaining
- $a$ is the original mass in grams
- $t$ is the time
- $x$ is the half-life of the isotope
Growth and Decay Factors

Identify each function as having a growth or decay factor.

\[ y = 1000(0.85)^x \quad f(x) = 0.5(3.3)^x \]

\[ b = 0.85 \quad b = 3.3 \]

\[ 0 < b < 1 \quad b > 1 \]

decay factor \quad growth factor

Write the growth or decay factor for each percent rate of change.

<table>
<thead>
<tr>
<th>Rate of Change</th>
<th>( r )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>45% increase</td>
<td>0.45</td>
<td>( 1 + r )</td>
</tr>
<tr>
<td>17.6% decrease</td>
<td>0.176</td>
<td>( 1 - r )</td>
</tr>
<tr>
<td>3.5% decrease</td>
<td>0.035</td>
<td>( 1 - 0.035 )</td>
</tr>
<tr>
<td>108% increase</td>
<td>1.08</td>
<td>( 1 + 1.08 )</td>
</tr>
</tbody>
</table>

Modeling Exponential Growth and Decay

A new home was purchased for $156,000 and the value of the home appreciates 2.5% each year. What is the value of the home after 5 years?

\[ y = ab^x \]

initial value: $156,000

\[ a = 156000 \]

percent rate of change: 2.5% \quad \[ r = 0.025 \]

\[ b = 1 + r \]
\[ b = 1 + 0.025 \]
\[ b = 1.025 \]

\[ y = 156000(1.025)^x \]
\[ y = 156000(1.025)^5 \]
\[ y = 176499.68 \]

After 5 years, the value of the home is $176,499.68.
Compound Interest

College Tuition The year he was born, Randall's grandparents deposited $5000 into an account that paid 9.5% interest annually. After 18 years, Randall will put the money toward his college tuition. How much money will be in the account for Randall's college tuition?

First, identify $P$, $r$, $n$, and $t$.

- Amount deposited: $5000
- Interest paid: 9.5%  
- Interest compounded: annually
- Number of years: 18

Next, substitute the values into the formula.

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$

Then, evaluate the expression.

$$A = 5000 \left(1 + \frac{0.095}{1}\right)^{1(18)}$$

After 18 years, there will be $25,610.86 in the account for Randall's college tuition.

Modeling Half-life

Carbon-14 The half-life of carbon-14 is 5,730 years. Write the exponential function that models the mass in grams $y$ of carbon-14 remaining after $t$ years in a rock that originally contained 157 grams. How many grams of carbon-14 will remain in the rock after 2,000 years?

$$y = 157(0.5)^{t/5730}$$

After 2,000 years, there will be **123.26 grams** of carbon-14 remaining in the rock.
Graphing One-Variable Data: Lesson Summary with Examples

Univariate data (or one-variable data) involves a single variable and allows you to describe something. There are two kinds of data.

- Categorical data is descriptive data that is not numeric. It can be observed, but not measured
- Quantitative data is numerical, which means it can be measured.

Frequency tables and relative frequency tables indicate how often individual events occur.

One-variable data can be represented with different kinds of graphs.

**Dot plots**
- Use dots to represent individual data points
- Represent frequencies for categorical or quantitative data
- Most effective for small data sets

**Histograms**
- Bar graphs that represent frequencies or relative frequencies
- Used for quantitative data
- Effective for any size data set

**Box plots**
- Represent five-number summaries for data sets
- Used for quantitative data
- Effective for any size data set
- May identify outliers
Quantitative data can be analyzed by calculating a five-number summary.

**Five-Number Summary**

- **Minimum**: smallest observed value that is not an outlier
- **Lower quartile (Q1)**: 25% of the data is at or below this value; 75% is at or above this value
- **Median**: the middle number; 50% of the data is at or below this value; 50% of the data is at or above this value
- **Upper quartile (Q3)**: 75% of the data is at or below this value; 25% is at or above this value
- **Maximum**: largest observed value that is not an outlier

Outliers are values much higher or lower than most of the values in a data set. To determine if a value is an outlier, calculate the interquartile range (the difference between the upper quartile (Q3) and the lower quartile (Q1)).

**Identifying Outliers**

- If a value is less than \( Q1 - 1.5(IQR) \), then it is an outlier.
- If a value is greater than \( Q3 + 1.5(IQR) \), then it is an outlier.
- A data set may have no outliers or may have multiple outliers.

**Types of Data**

**Data Types**

- Categorical data → qualitative → describes a quality
- Quantitative data → numerical → represents a quantity or number

The White House is located in **Washington, D.C.** Construction on the building ended in **1800**. The cost to build the **mansion** was **$232,372**. The exterior walls of the house were originally made of pale gray sandstone. The walls were later painted **white**. Today, the White House has **six floors** and **147 rooms**.
Frequency Tables

One way to organize both categorical and quantitative data is to use a frequency table. A frequency table displays how often individual events occur.

A relative frequency table shows the ratio of the frequency of an individual event to the total number of events. These are often expressed as percentages. By finding the relative frequencies, we can more easily understand the distribution of data.
To find the percentage of M&Ms that are either red or blue, we can add individual frequencies.

**Red or Blue M&M's**

- Percentage of red M&M's: 18%
- Percentage of blue M&M's: 21%
- Percentage of red or blue M&M's: $18\% + 21\% = 39\%$

By displaying the data using relative frequencies, we can also look at the probability of an event. Find the probability of selecting a green M&M at random.

$$P\text{(green)} = \frac{\text{number of green M&M's}}{\text{total number of M&M's}}$$

$$P\text{(green)} = \frac{10}{67}$$

**Dot Plots**

A dot plot is a graph where dots are used to represent individual data points. The dots are plotted above a number line. Dot plots can be used to represent frequencies for categorical or quantitative data.

To make a dot plot of this data, draw a horizontal axis labeled with each color. Then above each color, add dots to represent the number of M&M's of that color.
Histogram

When looking at quantitative data like this, if there is a lot of data or the data is spread out, it may not be possible to create a frequency table for every data value.

<table>
<thead>
<tr>
<th>State</th>
<th>Mean Math Score</th>
<th>State</th>
<th>Mean Math Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alabama</td>
<td>550</td>
<td>Montana</td>
<td>538</td>
</tr>
<tr>
<td>Alaska</td>
<td>515</td>
<td>Nebraska</td>
<td>593</td>
</tr>
<tr>
<td>Arizona</td>
<td>525</td>
<td>Nevada</td>
<td>501</td>
</tr>
<tr>
<td>Arkansas</td>
<td>566</td>
<td>New Hampshire</td>
<td>524</td>
</tr>
<tr>
<td>California</td>
<td>516</td>
<td>New Jersey</td>
<td>514</td>
</tr>
<tr>
<td>Colorado</td>
<td>572</td>
<td>New Mexico</td>
<td>549</td>
</tr>
<tr>
<td>Connecticut</td>
<td>514</td>
<td>New York</td>
<td>499</td>
</tr>
<tr>
<td>Delaware</td>
<td>495</td>
<td>North Carolina</td>
<td>511</td>
</tr>
<tr>
<td>District of Columbia</td>
<td>464</td>
<td>North Dakota</td>
<td>594</td>
</tr>
<tr>
<td>Florida</td>
<td>498</td>
<td>Ohio</td>
<td>548</td>
</tr>
<tr>
<td>Georgia</td>
<td>490</td>
<td>Oklahoma</td>
<td>568</td>
</tr>
<tr>
<td>Hawaii</td>
<td>505</td>
<td>Oregon</td>
<td>524</td>
</tr>
<tr>
<td>Idaho</td>
<td>541</td>
<td>Pennsylvania</td>
<td>501</td>
</tr>
<tr>
<td>Illinois</td>
<td>600</td>
<td>Rhode Island</td>
<td>495</td>
</tr>
<tr>
<td>Indiana</td>
<td>505</td>
<td>South Carolina</td>
<td>495</td>
</tr>
<tr>
<td>Iowa</td>
<td>613</td>
<td>South Dakota</td>
<td>603</td>
</tr>
<tr>
<td>Kansas</td>
<td>595</td>
<td>Tennessee</td>
<td>571</td>
</tr>
<tr>
<td>Kentucky</td>
<td>575</td>
<td>Texas</td>
<td>505</td>
</tr>
<tr>
<td>Louisiana</td>
<td>550</td>
<td>Utah</td>
<td>559</td>
</tr>
<tr>
<td>Maine</td>
<td>467</td>
<td>Vermont</td>
<td>521</td>
</tr>
<tr>
<td>Maryland</td>
<td>506</td>
<td>Virginia</td>
<td>512</td>
</tr>
<tr>
<td>Massachusetts</td>
<td>526</td>
<td>Washington</td>
<td>532</td>
</tr>
<tr>
<td>Michigan</td>
<td>605</td>
<td>West Virginia</td>
<td>507</td>
</tr>
<tr>
<td>Minnesota</td>
<td>607</td>
<td>Wisconsin</td>
<td>604</td>
</tr>
<tr>
<td>Mississippi</td>
<td>548</td>
<td>Wyoming</td>
<td>567</td>
</tr>
<tr>
<td>Missouri</td>
<td>595</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Instead, we group the data into equal sized intervals or classes. A histogram can be created to display the data.

### 2009-2010 Mean SAT Math Scores

<table>
<thead>
<tr>
<th>Score Intervals</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>460 – 479</td>
<td>2</td>
</tr>
<tr>
<td>480 – 499</td>
<td>6</td>
</tr>
<tr>
<td>500 – 519</td>
<td>13</td>
</tr>
<tr>
<td>520 – 539</td>
<td>7</td>
</tr>
<tr>
<td>540 – 559</td>
<td>7</td>
</tr>
<tr>
<td>560 – 579</td>
<td>6</td>
</tr>
<tr>
<td>580 – 599</td>
<td>4</td>
</tr>
<tr>
<td>600 – 619</td>
<td>6</td>
</tr>
</tbody>
</table>

### Key Characteristics of Histograms

- Intervals are labeled on the x-axis.
- Bar heights are labeled on the y-axis and represent the frequencies.
- Bars have the same width and are drawn next to each other with no gaps.
- Histograms typically contain 5–10 intervals (or classes).
## Five-Number Summary

The numbers that make up this summary are the minimum, lower quartile (Q1), median, upper quartile (Q3), and the maximum.

The median is the middle number in the data set. The lower quartile is the middle number for the numbers below the median. And the upper quartile is the middle number for the numbers above the median.

### Find the five-number summary

1. **Order the data from smallest to largest.**
   
   | 3 | 5 | 6 | 7 | 8 | 9 | 10 | 10 | 12 | 14 | 15 | 15 | 16 | 18 | 20 |

2. **Identify the minimum and the maximum values.**
   
   | 3 | 6 | 6 | 7 | 8 | 9 | 10 | 10 | 12 | 14 | 15 | 15 | 16 | 18 | 20 |

   - Minimum: 3
   - Maximum: 20

3. **Identify the median.**
   
   Find the middle number in the data set.
   
   | 3 | 6 | 6 | 7 | 8 | 9 | 10 | 12 | 14 | 15 | 15 | 16 | 18 | 20 |

   - Median: 10

4. **Find the lower quartile.**
   
   Find the middle number for the numbers below the median in the data set.
   
   | 3 | 6 | 6 | 7 | 8 | 9 | 10 | 12 | 14 | 15 | 15 | 16 | 18 | 20 |

   - Numbers below the median: 3, 6, 6, 7, 8, 9, 10
   - Lower quartile (Q1): 7

5. **Find the upper quartile.**
   
   Find the middle number for the numbers above the median in the data set.
   
   | 3 | 6 | 6 | 7 | 8 | 9 | 10 | 12 | 14 | 15 | 15 | 16 | 18 | 20 |

   - Numbers above the median: 12, 14, 15, 16, 18, 20
   - Upper quartile (Q3): 15
The word *quartile* actually comes from the Latin word meaning "fourth." By calculating the quartiles, you're dividing a data set into four equal groups. Each group contains *one-fourth* of the data.

\[
\begin{align*}
3 & \quad 6 & \quad 6 & \quad 7 & \quad 8 & \quad 9 & \quad 10 & \quad 12 & \quad 14 & \quad 15 & \quad 16 & \quad 18 & \quad 20 \\
\text{lower quartile} & \quad \text{median} & \quad \text{upper quartile} & \text{(Q1)} & \text{(Q3)}
\end{align*}
\]

Knowing the quartiles, we can describe the data in a number of ways. For example, for this data set:

- one-fourth (or 25%) of the data is less than or equal to 7
- three-quarters (or 75%) of the data is greater than or equal to 7
- one-half (or 50%) of the data is less than or equal to 10

If there are an even number of data points, find the median. Q1, or Q3 by finding the average of the two numbers in the middle.
Box Plot

A box plot provides a graphical representation of the five-number summary.

1. Draw a number line with equal intervals to cover the range of data (minimum to maximum).
2. Above the number line, plot each of the points in the five-number summary.
3. Draw a box that extends from the lower quartile (Q1) to the upper quartile (Q3).
4. Within the box, draw a vertical line to represent the median.
5. Add lines (whiskers) that extend to the minimum and maximum values.

Outliers

Within a data set, there may be one or more values that are much higher or lower than most of the data. These extreme values are called outliers.

The interquartile range (IQR) is the difference between the upper quartile (Q3) and the lower quartile (Q1) and is used to identify any outliers.

\[
\text{IQR} = Q_3 - Q_1 \\
\text{Outliers: Value } < Q_1 - 1.5\text{(IQR)} \quad \text{Value } > Q_3 + 1.5\text{(IQR)}
\]

Box plot:

Since 47 is an outlier, the maximum value is now 23, not 47.
Analyzing One-Variable Data: Lesson Summary with Examples

Measures of Center

- The mean is the sum of all the values in a data set, divided by the number of values. The mean should be used to describe the center when there are no outliers in the data set.
- The median is the middle number in an ordered set of data. The median should be used to describe the center when there are outliers in a data set.
- The mode is the value that appears most often in a data set. The mode should be used to describe data that is qualitative, not quantitative.

Shape

Data has a normal distribution or an approximately normal distribution if the graph is symmetric with a bell-shaped curve.

Data that is skewed left has less data on the left hand side and therefore the tail of the graph stretches out to the left.

Data that is skewed right has less data on the right hand side and therefore the tail of the graph stretches out to the right.
Measures of Spread

- Variability is the spread in the distribution of the values in a data set; it can tell us if the values in the data set are close together or spread apart.
- The range of a set of data is determined by subtracting the smallest value from the largest value. The range indicates the spread of the entire data set.
- The interquartile range (IQR) is the difference between the upper quartile (Q3) and the lower quartile (Q1). The IQR represents the middle half or 50% of the data.
- The standard deviation measures the amount by which individual terms in a data set differ from the mean. The greater the standard deviation, the more spread out the data set. The smaller the standard deviation, the less spread out the data set.

Finding Measures of Center

Find the measures of center for Peter's first 7 Algebra 1 tests: 88, 89, 92, 85, 94, 82, 94

<table>
<thead>
<tr>
<th>mean</th>
<th>median</th>
<th>mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>Add all values and divide by the number of values.</td>
<td>Organize the values in numerical order and identify the number in the middle.</td>
<td>Identify the value(s) that appear most often.</td>
</tr>
</tbody>
</table>
| \[
\frac{88 + 89 + 92 + 85 + 94 + 82 + 94}{7}
\] | 82, 85, 88, 89, 92, 94 | 82, 85, 88, 89, 92, 94, 94 |
| mean = 89.14 | median = 89 | mode = 94 |

The mean was most affected by the outlier of 0.
Understanding the Shape of a Graph

An outlier has a greater effect on the mean than it does on the median. When looking at skewed data, the mean is typically closer to the tail; it's in the direction of the skew.

**skewed left**

Data that is skewed left has *less data on the left-hand side*. The tail of the graph stretches out to the left.

![Graph](image)

The box plot has a longer whisker on the left-hand side.

The mean is less than the median.

**skewed right**

Data that is skewed right has *less data on the right-hand side*. The tail of the graph stretches out to the right.

![Graph](image)

The box plot has a longer whisker on the right-hand side.

The mean is greater than the median.
# Measuring Spread: Range and IQR

<table>
<thead>
<tr>
<th>School</th>
<th>Reading</th>
<th>Math</th>
<th>Writing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brown</td>
<td>650</td>
<td>670</td>
<td>670</td>
</tr>
<tr>
<td>Columbia</td>
<td>680</td>
<td>660</td>
<td>690</td>
</tr>
<tr>
<td>Cornell</td>
<td>630</td>
<td>660</td>
<td>-</td>
</tr>
<tr>
<td>Dartmouth</td>
<td>660</td>
<td>670</td>
<td>670</td>
</tr>
<tr>
<td>Harvard</td>
<td>690</td>
<td>690</td>
<td>690</td>
</tr>
<tr>
<td>Princeton</td>
<td>590</td>
<td>700</td>
<td>700</td>
</tr>
<tr>
<td>U Penn</td>
<td>660</td>
<td>690</td>
<td>670</td>
</tr>
<tr>
<td>Yale</td>
<td>700</td>
<td>700</td>
<td>700</td>
</tr>
</tbody>
</table>

## SAT Scores: Reading
- **range** = maximum – minimum
- **range** = 700 – 630
- **range** = 70
- **Q1** = 655
- **Q3** = 690
- **IQR** = 690 – 655
- **IQR** = 45

## SAT Scores: Math
- **range** = maximum – minimum
- **range** = 700 – 660
- **range** = 40
- **Q1** = 670
- **Q3** = 695
- **IQR** = 695 – 670
- **IQR** = 25

## SAT Scores: Writing
- **range** = maximum – minimum
- **range** = 700 – 670
- **range** = 30
- **Q1** = 670
- **Q3** = 700
- **IQR** = 700 – 670
- **IQR** = 30
Measuring Spread: Standard Deviation

The standard deviation measures the amount by which individual terms in a data set differ from the mean. In general, the larger the standard deviation, the more "spread out" the data set. And the smaller the standard deviation, the less "spread out" the data set. With a larger standard deviation, we expect to see a large range of values within the data set.

A basketball team is sponsored by a sporting goods store that provides shoes to each of the players. The team's shoe size is normally distributed with a mean of 11 and a standard deviation of 1.5.

- 68% of the shoe sizes are between 9.5 and 12.5
  \[11 - 1.5 = 9.5\]
  \[11 + 1.5 = 12.5\]
- 95% of the shoe sizes are between 8 and 14.
  \[11 - 2(1.5) = 8\]
  \[11 + 2(1.5) = 14\]
- 99% of the shoe sizes are between 6.5 and 15.5.
  \[11 - 3(1.5) = 6.5\]
  \[11 + 3(1.5) = 15.5\]
Comparing Data Sets

The box plots represent the average monthly temperatures (in °F) for four U.S. cities.

- City D has the smallest range of average monthly temperatures.
- City A has the greatest range of average monthly temperatures.
- City D has the smallest interquartile range.
- City C has the highest median.
Two-Variable Categorical Data: Lesson Summary with Examples

A two-way frequency table is a useful way to display data in order to examine the relationship between two categorical variables. The rows represent the categories of one variable and the columns represent the categories of the other variable. Each entry in this type of table represents a frequency count (how often an event occurs). Marginal frequencies are the entries in the total row and column. Joint frequencies are the entries in the body of the table.

<table>
<thead>
<tr>
<th>Favorite Ice Cream</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chocolate</td>
</tr>
<tr>
<td>Men</td>
</tr>
<tr>
<td>Women</td>
</tr>
<tr>
<td>Total</td>
</tr>
</tbody>
</table>

A two-way relative frequency table displays the relative frequencies for each entry as compared to the total number of observations.

<table>
<thead>
<tr>
<th>Chocolate</th>
<th>Vanilla</th>
<th>Strawberry</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Men</td>
<td>0.128</td>
<td>0.200</td>
<td>0.256</td>
</tr>
<tr>
<td>Women</td>
<td>0.216</td>
<td>0.120</td>
<td>0.080</td>
</tr>
<tr>
<td>Total</td>
<td>0.344</td>
<td>0.320</td>
<td>0.336</td>
</tr>
</tbody>
</table>

A relative frequency table by row or by column compares data within a specific category, rather than against the total number of observations.

**Relative Frequency by Row**

<table>
<thead>
<tr>
<th>Chocolate</th>
<th>Vanilla</th>
<th>Strawberry</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Men</td>
<td>0.219</td>
<td>0.342</td>
<td>0.438</td>
</tr>
<tr>
<td>Women</td>
<td>0.519</td>
<td>0.288</td>
<td>0.192</td>
</tr>
<tr>
<td>Total</td>
<td>0.344</td>
<td>0.320</td>
<td>0.336</td>
</tr>
</tbody>
</table>

**Relative Frequency by Column**

<table>
<thead>
<tr>
<th>Chocolate</th>
<th>Vanilla</th>
<th>Strawberry</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Men</td>
<td>0.372</td>
<td>0.625</td>
<td>0.762</td>
</tr>
<tr>
<td>Women</td>
<td>0.628</td>
<td>0.375</td>
<td>0.238</td>
</tr>
<tr>
<td>Total</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>
Two-Way Frequency Table

A two-way frequency table is a table that displays data in order to examine the relationship between two categorical variables.

The rows represent the categories of one variable and the columns represent the categories of the other variable. Each entry in this type of table represents a frequency count (how often an event occurs).

The joint frequency refers to the entries in the body of a two-way frequency table. The marginal frequency refers to the entries in the "total" row and the "total" column of a two-way frequency table.

We can use the joint and marginal frequencies to analyze the data.

- Of those surveyed, 4 were females who drove to work.
- 11 of those surveyed drove to work.
- 10 of those surveyed were male.
- More men were surveyed than women.
Two-Way Relative Frequency Table

Two-variable categorical data can be displayed in a two-way relative frequency table where each entry represents the frequency of an individual event relative to all observations in the table.

Ice Cream Survey 125 people were asked to name their favorite flavor of ice cream.

<table>
<thead>
<tr>
<th></th>
<th>Chocolate</th>
<th>Vanilla</th>
<th>Strawberry</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Men</td>
<td>16</td>
<td>25</td>
<td>32</td>
<td>73</td>
</tr>
<tr>
<td>Women</td>
<td>27</td>
<td>15</td>
<td>10</td>
<td>52</td>
</tr>
<tr>
<td>Total</td>
<td>43</td>
<td>40</td>
<td>42</td>
<td>125</td>
</tr>
</tbody>
</table>

To display the data in a two-way relative frequency table, divide each table entry by 125, the total number surveyed. The values can be expressed as ratios, decimals, or percentages.

Ratios

<table>
<thead>
<tr>
<th></th>
<th>Chocolate</th>
<th>Vanilla</th>
<th>Strawberry</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Men</td>
<td>(\frac{16}{125})</td>
<td>(\frac{25}{125})</td>
<td>(\frac{32}{125})</td>
<td>(\frac{73}{125})</td>
</tr>
<tr>
<td>Women</td>
<td>(\frac{27}{125})</td>
<td>(\frac{15}{125})</td>
<td>(\frac{10}{125})</td>
<td>(\frac{52}{125})</td>
</tr>
<tr>
<td>Total</td>
<td>(\frac{43}{125})</td>
<td>(\frac{40}{125})</td>
<td>(\frac{42}{125})</td>
<td>(\frac{125}{125})</td>
</tr>
</tbody>
</table>

Decimals

<table>
<thead>
<tr>
<th></th>
<th>Chocolate</th>
<th>Vanilla</th>
<th>Strawberry</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Men</td>
<td>0.128</td>
<td>0.200</td>
<td>0.256</td>
<td>0.584</td>
</tr>
<tr>
<td>Women</td>
<td>0.216</td>
<td>0.120</td>
<td>0.080</td>
<td>0.416</td>
</tr>
<tr>
<td>Total</td>
<td>0.344</td>
<td>0.320</td>
<td>0.336</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Percentages

<table>
<thead>
<tr>
<th></th>
<th>Chocolate</th>
<th>Vanilla</th>
<th>Strawberry</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Men</td>
<td>12.6%</td>
<td>20.0%</td>
<td>25.6%</td>
<td>58.4%</td>
</tr>
<tr>
<td>Women</td>
<td>21.6%</td>
<td>12.0%</td>
<td>6.0%</td>
<td>41.6%</td>
</tr>
<tr>
<td>Total</td>
<td>34.4%</td>
<td>32.0%</td>
<td>33.6%</td>
<td>100.0%</td>
</tr>
</tbody>
</table>

When interpreting a two-way relative frequency table, remember that each entry is being compared to the total number of observations. It is common to use phrases like "What percent of the total..." or "Of the number surveyed..." when interpreting relative frequencies. We may also look at each value as the intersection of two categories.

For example, 12.8% of the people surveyed are men who prefer chocolate ice cream.
Relative Frequency by Row and by Column

Another way to look at data is to compare values within a specific category rather than against the total number of observations. These tables display the relative frequency by row or by column where each entry represents the frequency of an individual event relative to the total number of observations for the row or the column. These two different types of relative frequency tables allow us to look at different relationships within the data.

**Favorite Activity Survey** 100 adults were asked their favorite recreational activity: shopping, going to the movies, or playing sports.

### Relative Frequency Table by Row

To complete the relative frequency table by row, find the total number of observations for each row. Then divide each row entry by the total number of observations for that row.

<table>
<thead>
<tr>
<th></th>
<th>Shopping</th>
<th>Movies</th>
<th>Sports</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>2</td>
<td>13</td>
<td>25</td>
<td>40</td>
</tr>
<tr>
<td>Female</td>
<td>33</td>
<td>8</td>
<td>19</td>
<td>60</td>
</tr>
<tr>
<td>Total</td>
<td>35</td>
<td>21</td>
<td>44</td>
<td>100</td>
</tr>
</tbody>
</table>

**Identify the total number of observations for each row.**

<table>
<thead>
<tr>
<th></th>
<th>Shopping</th>
<th>Movies</th>
<th>Sports</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>2/40</td>
<td>13/40</td>
<td>25/40</td>
<td>40/40</td>
</tr>
<tr>
<td>Female</td>
<td>33/60</td>
<td>8/60</td>
<td>19/60</td>
<td>60/60</td>
</tr>
<tr>
<td>Total</td>
<td>35/100</td>
<td>21/100</td>
<td>44/100</td>
<td>100/100</td>
</tr>
</tbody>
</table>

**Divide all row entries by the total number of observations for the row.**

<table>
<thead>
<tr>
<th></th>
<th>Shopping</th>
<th>Movies</th>
<th>Sports</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>0.05</td>
<td>0.325</td>
<td>0.625</td>
<td>1.00</td>
</tr>
<tr>
<td>Female</td>
<td>0.55</td>
<td>0.133</td>
<td>0.317</td>
<td>1.00</td>
</tr>
<tr>
<td>Total</td>
<td>0.35</td>
<td>0.21</td>
<td>0.44</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Relative Frequency Table by Column

To complete the relative frequency table by column, find the total number of observations for each column.

<table>
<thead>
<tr>
<th></th>
<th>Shopping</th>
<th>Movies</th>
<th>Sports</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>2</td>
<td>13</td>
<td>25</td>
<td>40</td>
</tr>
<tr>
<td>Female</td>
<td>33</td>
<td>8</td>
<td>19</td>
<td>60</td>
</tr>
<tr>
<td>Total</td>
<td>35</td>
<td>21</td>
<td>44</td>
<td>100</td>
</tr>
</tbody>
</table>

Then divide each column entry by the total number of observations for that column.

<table>
<thead>
<tr>
<th></th>
<th>Shopping</th>
<th>Movies</th>
<th>Sports</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>(\frac{2}{35})</td>
<td>(\frac{13}{21})</td>
<td>(\frac{25}{44})</td>
<td>(\frac{40}{100})</td>
</tr>
<tr>
<td>Female</td>
<td>(\frac{33}{55})</td>
<td>(\frac{8}{22})</td>
<td>(\frac{19}{44})</td>
<td>(\frac{50}{100})</td>
</tr>
<tr>
<td>Total</td>
<td>(\frac{35}{55})</td>
<td>(\frac{21}{22})</td>
<td>(\frac{44}{44})</td>
<td>(\frac{100}{100})</td>
</tr>
</tbody>
</table>

When interpreting relative frequencies by row or column, keep in mind that each entry is being compared to the total number of observations in its row or column. Therefore, it is common to use phrases that refer to the subcategories, like "what percent of the men" or "of the group surveyed who like shopping", when interpreting these relative frequencies.

Examples:

- The statement, "Of those who preferred going to the movies, 38.1% were female" interprets the relative frequency by column.
- The statement, "62.5% of males that preferred playing sports" interprets the relative frequency by row.
Two-Variable Quantitative Data: Lesson Summary with Examples

Bivariate data involves two quantitative variables and deals with relationships between those variables. By plotting bivariate data as ordered pairs, we can create a scatterplot. This type of graph allows us to visually compare data and identify possible relationships between variables.

Correlation is a relationship between two variables in a scatterplot. The correlation can be positive, negative, or there may not be a correlation.

Scatterplots can reveal patterns or trends in bivariate data.

If data exhibits a **linear trend**, points within the scatterplot will generally gather around a line. The values increase or decrease at a fairly constant rate of change.

If data exhibits an **exponential trend**, points within the scatterplot will generally gather around an exponential curve. The values increase or decrease at a fairly constant *percentage* rate of change, or common ratio.

The line or curve of best fit is an equation that models the relationship between the variables. The line of best fit or linear regression can be used to make predictions.

The correlation coefficient and residuals can be used to determine if a linear regression model is a good fit for a data set.
The closer \( r \) is to 1 or \(-1\), the more closely the data points will cluster around the line of best fit, demonstrating how well the linear regression models the data.

A residual is the difference between the observed \( y \)-value (from the scatterplot) and the predicted \( y \)-value (from the line of best fit).

A residual plot is a scatterplot of all the residuals for a data set. Residual plots are used to determine whether a regression model is a good fit (residual plot is random) or not a good fit (residual plot is non-random) for the bivariate data.

Causation is a relationship in which the change in the value of one variable causes a change in the value of the other variable.

**Scatterplots**

To graph a scatterplot of bivariate data, identify which is the independent and which is the dependent variable. Then plot each point on a coordinate plane.

This scatterplot shows that there is a relationship between the amount of time spent studying and the SAT math scores.
Identifying Trends

This table shows the relationship between temperature and solubility for potassium nitrate (KNO₃).

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>Solubility (g per 100 g H₂O)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td>10</td>
<td>22</td>
</tr>
<tr>
<td>20</td>
<td>32</td>
</tr>
<tr>
<td>30</td>
<td>48</td>
</tr>
<tr>
<td>40</td>
<td>65</td>
</tr>
<tr>
<td>50</td>
<td>85</td>
</tr>
<tr>
<td>60</td>
<td>108</td>
</tr>
<tr>
<td>70</td>
<td>137</td>
</tr>
</tbody>
</table>

This data appears to have an exponential trend.

This table shows how the wavelength of light changes when the light passes from air to water.

<table>
<thead>
<tr>
<th>Wavelength in Air (nm)</th>
<th>Wavelength in Water (nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>475</td>
<td>357</td>
</tr>
<tr>
<td>490</td>
<td>368</td>
</tr>
<tr>
<td>505</td>
<td>380</td>
</tr>
<tr>
<td>520</td>
<td>391</td>
</tr>
<tr>
<td>535</td>
<td>402</td>
</tr>
<tr>
<td>550</td>
<td>414</td>
</tr>
<tr>
<td>565</td>
<td>425</td>
</tr>
<tr>
<td>580</td>
<td>436</td>
</tr>
<tr>
<td>595</td>
<td>447</td>
</tr>
<tr>
<td>610</td>
<td>459</td>
</tr>
<tr>
<td>625</td>
<td>470</td>
</tr>
</tbody>
</table>

This data appears to have a linear trend.
Understanding Best Fit

**Median Income** The table and the scatterplot below show the median yearly income of U.S. males in the years from 1950 to 2007.

<table>
<thead>
<tr>
<th>Years Since 1950</th>
<th>Median Yearly Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2570</td>
</tr>
<tr>
<td>10</td>
<td>4080</td>
</tr>
<tr>
<td>20</td>
<td>6670</td>
</tr>
<tr>
<td>30</td>
<td>12530</td>
</tr>
<tr>
<td>40</td>
<td>20293</td>
</tr>
<tr>
<td>50</td>
<td>28343</td>
</tr>
<tr>
<td>57</td>
<td>33195</td>
</tr>
</tbody>
</table>

There are two possible models for the data:

- **Linear**
  
  The linear model passes through 1 of the points \((10, 4080)\) while the others points fall either above or below the line.

- **Exponential**
  
  The exponential model appears to pass through four points: \((0, 2570)\), \((10, 4080)\), \((20, 6670)\), and \((50, 28343)\). While some points do fall slightly above or below the curve, they are closer to the curve than the points for the linear model. So the exponential curve appears to be the better fit.
The best fit model can be used to make predictions.

**Babies Weight** The following table represents the median expected weights for girls from birth to six months of age.

<table>
<thead>
<tr>
<th>Age (months)</th>
<th>Weight (pounds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7.0</td>
</tr>
<tr>
<td>1</td>
<td>9.2</td>
</tr>
<tr>
<td>2</td>
<td>11.2</td>
</tr>
<tr>
<td>3</td>
<td>12.8</td>
</tr>
<tr>
<td>4</td>
<td>14.1</td>
</tr>
<tr>
<td>5</td>
<td>15.2</td>
</tr>
<tr>
<td>6</td>
<td>16.1</td>
</tr>
</tbody>
</table>

Locate the $y$-value that corresponds to the $x$-value of 8. At 8 months, the expected median weight for a girl is about 20 pounds.

Using the best fit model, substitute 8 into the equation for $x$ and solve for $y$.

\[
y = 1.51x + 7.71
\]

\[
y = 1.51(8) + 7.71
\]

\[
y = 12.08 + 7.71
\]

\[
y = 19.79
\]

So the median expected weight for an 8 month old girl is 19.79 pounds.

**Linear Regression Models**

Another name for a model of best fit is a regression model. One way to find the linear regression is to use a graphing calculator. Using the linear regression model, we can interpret the slope and $y$-intercept.

\[
y = 25x + 317
\]

\[
\text{slope} = \frac{\text{change in SAT score}}{\text{change in hours}} = \frac{29 \text{ points}}{1 \text{ hour}}
\]

The SAT score increases 29 points for each hour spent studying. If a student spends no time studying, the SAT score is 317.
Correlation Coefficients

The strength of a linear regression is reflected by the value of the correlation coefficient $r$.

When the correlation coefficient is $-1$, the linear regression models the data perfectly.

When the correlation coefficient is $-0.9$, the linear regression is a strong model for the data.

When the correlation coefficient is $-0.5$, the linear regression is a moderately strong model for the data.

When the correlation coefficient is $0$, the linear regression does not closely model the data.

When the correlation coefficient is $0.5$, the linear regression is a moderately strong model for the data.

When the correlation coefficient is $0.9$, the linear regression is a strong model for the data.
When the correlation coefficient is \(-1\), the linear regression models the data perfectly.

**Understanding Residuals**

A residual is the difference between the observed \(y\)-value (from the scatterplot) and the predicted \(y\)-value (from the line of best fit).

To find the predicted values, substitute the \(x\)-values of each point into the linear regression. For example, to find the corresponding predicted \(y\)-value using the point (0, 3.6), substitute \(x = 0\) into \(y = -0.13x + 3.56\). The predicted value is 3.56.

Calculate the residual for each data point.

**Residuals for Data Points**

residual = observed \(y\)-value minus predicted \(y\)-value

- Point 1: (0, 3.6) \(\quad 3.6 - 3.56 = 0.04\)
- Point 2: (1, 3.4) \(\quad 3.4 - 3.43 = -0.03\)
- Point 3: (2, 3.2) \(\quad 3.2 - 3.3 = -0.10\)
- Point 4: (3, 3.3) \(\quad 3.3 - 3.17 = 0.13\)
- Point 5: (4, 3.0) \(\quad 3.0 - 3.04 = -0.04\)
- Point 6: (5, 2.9) \(\quad 2.9 - 2.91 = -0.01\)

A positive residual indicates that the observed value is above the line of best fit.

A negative residual indicates that the observed value is below the line of best fit.

If we found the residual was equal to 0, the observed value would be on the line of best fit. In other words, the observed and predicted values would be the same.
A residual plot is a scatterplot of all the residuals for a data set. Residual plots are used to determine whether a regression model is a good fit for the bivariate data.

A residual plot that is random, with points dispersed around the horizontal axis, indicates that the regression model is a good fit for the data.

Non-random residual plots indicate that the chosen regression models are not good fits for the data.

To create a residual plot, plot each ordered pair \((x, \text{residual})\) using the \(x\)-value of the data set and the corresponding residual value.

<table>
<thead>
<tr>
<th>Hours Spent Watching TV (per day)</th>
<th>Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.04</td>
</tr>
<tr>
<td>1</td>
<td>-0.03</td>
</tr>
<tr>
<td>2</td>
<td>-0.10</td>
</tr>
<tr>
<td>3</td>
<td>0.13</td>
</tr>
<tr>
<td>4</td>
<td>-0.04</td>
</tr>
<tr>
<td>5</td>
<td>-0.01</td>
</tr>
</tbody>
</table>
Simplifying Radicals: Lesson Summary with Examples

A radical expression is an expression that contains a radical sign. The index states the type of root of the radical expression. The radicand is the number or expression under the radical sign.

\[
\text{index} \rightarrow n\sqrt{b} \leftarrow \text{radicand}
\]

A square root is in simplest form when
- The radicand has no perfect square factors other than 1.
- The radicand is not a fraction.
- A radical is not in the denominator of a fraction.

Radical Expressions

Examples of radical expression

<table>
<thead>
<tr>
<th>square root</th>
<th>cube root</th>
<th>4th root</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{3} )</td>
<td>( \frac{3}{3} \sqrt{27} )</td>
<td>( \frac{4}{4} \sqrt{65} )</td>
</tr>
<tr>
<td>( \sqrt{18x} )</td>
<td>( \frac{3}{3} \sqrt{54x^3y^5} )</td>
<td>( \frac{4}{4} \sqrt{9x^2yz^3} )</td>
</tr>
<tr>
<td>10( \sqrt{7} )</td>
<td>5( \sqrt{4} )</td>
<td>( -3\sqrt{106} )</td>
</tr>
</tbody>
</table>

To rationalize the denominator rewrite the radical expression so there is no radical in the denominator by multiplying the numerator and denominator.
Prime Factorization

Find the prime factorization of 18.

Prime factorization: $18 = 2 \cdot 3 \cdot 3$

Simplifying Square Roots

Square roots can be simplified in two ways.

Simplify by identifying perfect squares.

\[
\begin{align*}
\sqrt{300} &= \sqrt{100 \cdot 3} \\
&= 10 \cdot \sqrt{3} \\
&= 10\sqrt{3}
\end{align*}
\]

Simplify by identifying common pairs.

\[
\begin{align*}
\sqrt{300} &= \sqrt{2 \cdot 2 \cdot 3 \cdot 5 \cdot 5} \\
&= 2 \cdot 5 \cdot \sqrt{3}
\end{align*}
\]

Simplify $\sqrt{75y^5}$.

\[
\begin{align*}
\sqrt{75y^5} &= \sqrt{3 \cdot 5 \cdot 5 \cdot y \cdot y \cdot y \cdot y} \\
&= 5 \cdot y \cdot y \cdot \sqrt{3 \cdot y} \\
&= 5y^2 \cdot \sqrt{3y}
\end{align*}
\]
### Multiplying Square Roots

Simplify each radical expression.

<table>
<thead>
<tr>
<th>( \sqrt{3} \cdot \sqrt{2} )</th>
<th>( \sqrt{6} \cdot \sqrt{3} )</th>
<th>( 2\sqrt{5} \cdot 4\sqrt{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{3} \cdot 2 )</td>
<td>( \sqrt{6} \cdot 3 )</td>
<td>( 2 \cdot 4\sqrt{5} \cdot 2 )</td>
</tr>
<tr>
<td>( \sqrt{6} )</td>
<td>( \sqrt{2} \cdot 3 \cdot 3 )</td>
<td>( 8\sqrt{10} )</td>
</tr>
</tbody>
</table>

| \( 3\sqrt{2} \) |}

### Dividing Square Roots

Simplify each radical expression.

<table>
<thead>
<tr>
<th>( \sqrt{25} )</th>
<th>( \sqrt{36} )</th>
<th>( \sqrt{44} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{25} )</td>
<td>( \sqrt{36} )</td>
<td>( \sqrt{y^4} )</td>
</tr>
<tr>
<td>( \sqrt{25} )</td>
<td>( \sqrt{36} )</td>
<td>( \sqrt{y^4} )</td>
</tr>
<tr>
<td>( 5 )</td>
<td>( 3\sqrt{5} )</td>
<td>( 2\sqrt{11} )</td>
</tr>
<tr>
<td>( 6 )</td>
<td>( 8 )</td>
<td>( \sqrt{y^2} )</td>
</tr>
</tbody>
</table>

### Rationalizing the Denominator

Simplify \( \frac{2}{\sqrt{3}} \).

\[
\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}
\]

\[
\frac{2\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}
\]

\[
\frac{2\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}
\]

\[
\frac{2\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}
\]

\[
\frac{2\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}
\]

\[
\frac{2\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}
\]
Simplifying $n^{th}$ Roots

$n^{th}$ roots, such as cube roots can be simplified in two ways.

Simplify by identifying common groups of 3.

\[ \sqrt[3]{40} \]
\[ = \sqrt[3]{2 \cdot 2 \cdot 5} \]
\[ = 2 \cdot \sqrt[3]{5} \]
\[ = 2\sqrt[3]{5} \]

Simplify by identifying perfect cubes.

\[ \sqrt[3]{40} \]
\[ = \sqrt[3]{8 \cdot 5} \]
\[ = 2 \cdot \sqrt[3]{5} \]
\[ = 2\sqrt[3]{5} \]

Simplify $\sqrt[4]{243a^6}$.

\[ \sqrt[4]{243a^6} \]
\[ = \sqrt[4]{3 \cdot 3 \cdot 3 \cdot 3 \cdot a \cdot a \cdot a \cdot a} \]
\[ = 3 \cdot a \cdot \sqrt[4]{3 \cdot a \cdot a} \]
\[ = 3a\sqrt[4]{3a^2} \]
Operations with Radicals: Lesson Summary with Examples

Radical expressions which have the same radicand and the same index are called like radicals. Simplify like radicals by adding or subtracting coefficients.

Like Radicals

Examples of like and unlike radicals

<table>
<thead>
<tr>
<th>like radicals</th>
<th>unlike radicals</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{3} ) and ( 4\sqrt{3} )</td>
<td>( \frac{3}{\sqrt{3}} ) and ( 4\sqrt{3} )</td>
</tr>
<tr>
<td>( 6\sqrt{7} ) and ( -\frac{3}{\sqrt{7}} )</td>
<td>( \frac{\sqrt{5}}{5} ) and ( \sqrt{\frac{4}{5}} )</td>
</tr>
<tr>
<td>( 5\sqrt{22} ), ( -8\sqrt{22} ), and ( 9\sqrt{22} )</td>
<td>( 5\sqrt{2} ), ( \sqrt{11} ), and ( \sqrt{2} )</td>
</tr>
</tbody>
</table>

Combining Like Radicals

Simplify \( 11\sqrt{5} + 4\sqrt{5} \).

- \( 11\sqrt{5} + 4\sqrt{5} \)
- \( 11\sqrt{5} + 4\sqrt{5} \)
- \( (11 + 4)\sqrt{5} \)
- \( 15\sqrt{5} \)

Simplify \( -8\sqrt{6} - 2\sqrt{6} \).

- \( -8\sqrt{6} - 2\sqrt{6} \)
- \( -8\sqrt{6} - 2\sqrt{6} \)
- \( (-8 - 2)\sqrt{6} \)
- \( -10\sqrt{6} \)

Simplify \( 8\sqrt{10} + \frac{5\sqrt{4}}{2} - 2\sqrt{10} + 7\sqrt{4} \).

- \( 8\sqrt{10} + \frac{5\sqrt{4}}{2} - 2\sqrt{10} + 7\sqrt{4} \)
- \( 8\sqrt{10} + (1 + 7)\sqrt{4} \)
- \( 6\sqrt{10} + 8\sqrt{4} \)

Identify the like radicals.

Add and subtract the coefficients of the like radicals.
Combining Radicals with Unlike Radicands

Simplify $\sqrt{54} + \sqrt{24}$.

$$\begin{align*}
\sqrt{54} + \sqrt{24} &= \sqrt{9 \cdot 6} + \sqrt{4 \cdot 6} \\
&= 3\sqrt{6} + 2\sqrt{6} \\
&= 5\sqrt{6}
\end{align*}$$

Simplifying Radicals Using the Distributive Property

Simplify $\sqrt{5}(2 + \sqrt{3})$.

$$\begin{align*}
\sqrt{5}(2 + \sqrt{3}) &= \sqrt{5} \cdot 2 + \sqrt{5} \cdot \sqrt{3} \\
&= 2\sqrt{5} + \sqrt{15}
\end{align*}$$
Rational Exponents: Lesson Summary with Examples

A rational exponent is an exponent written as a fraction.

Expressions with rational exponents can be written in radical form. Radical expressions can be written using rational exponents.

### Rational Exponents

#### Properties of Exponents

<table>
<thead>
<tr>
<th>Property</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product of Powers</td>
<td>(a^m \cdot a^n = a^{m+n})</td>
</tr>
<tr>
<td>Quotient of Powers</td>
<td>(\frac{a^m}{a^n} = a^{m-n})</td>
</tr>
<tr>
<td>Power of a Power</td>
<td>((a^m)^n = a^{mn})</td>
</tr>
<tr>
<td>Power of a Product</td>
<td>((ab)^m = a^m b^m)</td>
</tr>
</tbody>
</table>

When multiplying like bases, add the exponents.

When dividing like bases, subtract the exponents.

When raising a power to a power, multiply the exponents.

When raising a product to a power, raise each factor to the power.

#### Rational Exponents

For all numbers \(a\) where \(a \geq 0\) and positive integers \(n\) where \(n \neq 1\),

\[a^\frac{1}{n} = \sqrt[n]{a}\]

For all numbers \(a\) where \(a \geq 0\) and positive integers \(m\) and \(n\) where \(n \neq 1\),

\[a^\frac{m}{n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m\]
**Properties of Exponents**

### Power of a Quotient

\[
\left( \frac{a}{b} \right)^m = \frac{a^m}{b^m}
\]

When raising a quotient to a power, raise both the numerator and the denominator to the power.

**Example:**

\[
\left( \frac{x^2}{y^3} \right)^3 = \frac{x^6}{y^9}
\]

### Zero Exponent Property

\[a^0 = 1\]

When raising a base to the zero power, the result is 1.

**Example:**

\[
\left( \frac{2}{x^3y} \right)^0 = 1
\]

### Rational Exponents as Radicals

Write each expression in radical form.

**Example 1**

\[
\begin{align*}
\frac{1}{a^n} &= \sqrt[n]{a} \\
x^{\frac{1}{2}} &= \sqrt{x} \\
\sqrt[3]{x} &= \sqrt[3]{x}
\end{align*}
\]

**Example 2**

\[
\begin{align*}
\frac{m}{a^n} &= \sqrt[n]{a^m} \\
2^{\frac{3}{4}} &= \sqrt[4]{2^3} \\
\frac{1}{\sqrt[3]{x}} &= \sqrt[3]{\frac{1}{x}}
\end{align*}
\]

### Radicals as Rational Exponents

Write the expression in simplest radical form.

\[
\begin{align*}
\frac{6}{\sqrt{x}} &= 6x^{-\frac{1}{2}} \\
\frac{2}{x^{\frac{5}{3}}} &= 2x^{-\frac{5}{3}} \\
\frac{1}{x^{\frac{2}{3}}} &= x^{-\frac{2}{3}} \\
\frac{3}{\sqrt[3]{x}} &= 3x^{-\frac{1}{3}}
\end{align*}
\]
Polynomials: Lesson Summary with Examples

A monomial is a number, a variable, or the product of a number and one or more variables with nonnegative integer exponents. A binomial is a polynomial with two terms. A trinomial is a polynomial with three terms.

A polynomial whose terms are written in descending order by degree is written in standard form.

### Classifying Polynomials by Degree

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Degree</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-19)</td>
<td>0</td>
<td>constant</td>
</tr>
<tr>
<td>(-6x + 1)</td>
<td>1</td>
<td>linear</td>
</tr>
<tr>
<td>(\frac{2}{3}x^2 + 4x - 8)</td>
<td>2</td>
<td>quadratic</td>
</tr>
<tr>
<td>(-2x^3 + 6x^2 + 4x - 5)</td>
<td>3</td>
<td>cubic</td>
</tr>
</tbody>
</table>

### Classifying Polynomials by Number of Terms

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Number of Terms</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-19)</td>
<td>1</td>
<td>monomial</td>
</tr>
<tr>
<td>(-6x + 1)</td>
<td>2</td>
<td>binomial</td>
</tr>
<tr>
<td>(\frac{2}{3}x^2 + 4x - 8)</td>
<td>3</td>
<td>trinomial</td>
</tr>
<tr>
<td>(-2x^3 + 6x^2 + 4x - 5)</td>
<td>4</td>
<td>polynomial</td>
</tr>
</tbody>
</table>

To add polynomials

1. Identify and group like terms.
2. Simplify

To subtract polynomials

1. Write the expression using addition.
2. Identify like terms and simplify.

### Monomials

The degree of a monomial with one variable is the exponent of the variable. The degree of a monomial with multiple variables is the sum of the exponents of the variables.

- \(3a^2\)  
  The exponent of the variable is 2, so \(3a^2\) has a degree of 2.

- \(-8x^2yz^5\)  
  The exponents of the variables are 2, 1 and 5.
  \[2 + 1 + 5 = 8\]  
  So \(-8x^2yz^5\) has a degree of 8.

- \(6^{12}m^2n^4\)  
  The exponents of the variables are 2 and 4.
  \[2 + 4 = 6\]  
  So \(6^{12}m^2n^4\) has a degree of 6.
Classifying Polynomials

Find the degree of the polynomial \(3x^2 + 5x - 4\).

Identify the terms in the polynomial. \(3x^2\) degree: 2
Find the degree of each term. \(5x\) degree: 1
The highest degree of the terms is the degree of the polynomial. \(-4\) degree: 0

The polynomial \(3x^2 + 5x - 4\) has a degree of 2.

Simplifying Polynomial Expressions

Simplify \(6x^2 - 8x^2 + 24x + 9x^3 - 7x + x^2\).

1. Identify like terms. \(6x^2 - 8x^2 + 24x + 9x^3 - 7x + x^2\)
2. Combine like terms. \(6x^2 - 8x^2 + 24x + 9x^3 - 7x + x^2\)

\(6x^2 + x^2 - 8x^3 + 9x^3 + 24x - 7x\)
\(7x^2 + x^3 + 17x\)
\(x^3 + 7x^2 + 17x\)

Adding Polynomials

Polynomials can be added horizontally or vertically. Adding polynomials horizontally is similar to simplifying polynomials – identify the like terms, then combine them.

Add the polynomials.
\((12x - 4) + (-6x + 11) + (2x^2 + x + 5) + (x^2 + 2)\)
1. Identify and group like terms. \((12x - 6x + x) + (2x^2 + x^2) + (-4 + 11 + 5 + 2)\)
2. Simplify. \(3x^2 + 7x + 14\)

To add polynomials vertically, align like terms.
Add \((x^2 + 6x - 4)\) and \((9x^2 + x - 15)\).

1. Align like terms. \(x^2 + 6x - 4\)
2. Simplify. \(9x^2 + x - 15\)

\(10x^2 + 7x - 19\)
Subtracting Polynomials

Subtracting polynomials is very similar to adding polynomials. We can subtract polynomials using either the vertical method or the horizontal method.

\[
\begin{align*}
\text{vertical method} \\
-4x^2 + 2x + 1 \\
- (6x^2 - x + 3) \\
\hline
-10x^2 + 3x - 2
\end{align*}
\]

\[
\begin{align*}
\text{horizontal method} \\
(-4x^2 + 2x + 1) - (6x^2 - x + 3) \\
-4x^2 + 2x + 1 - 6x^2 + x - 3 \\
(-4x^2 - 6x^2) + (2x + x) + (1 - 3) \\
-10x^2 + 3x - 2
\end{align*}
\]
Multiplying Polynomials: *Lesson Summary with Examples*

To multiply polynomials
1. Distribute each term in the first polynomial to each term in the second polynomial.
2. Simplify by combining like terms.

### Multiplying Binomials Using the FOIL Method

To multiply two binomials, multiply the First terms, the Outer terms, the Inner terms, and the Last terms.

\[(x - 3)(x + 5) = x^2 + 5x - 3x - 15\]
\[= x^2 + 2x - 15\]

### Square of a Binomial

\[(a + b)^2 = a^2 + 2ab + b^2\]
\[(a - b)^2 = a^2 - 2ab + b^2\]

### Product of a Sum and a Difference

\[(a + b)(a - b) = a^2 - ab + ab - b^2\]
\[= a^2 - b^2\]
Multiplying with Monomials

Multiply $3x$ and $(2x + 1)$ using algebra tiles.

$$3x(2x + 1)$$

We can use the Distributive Property to multiply monomials and polynomials algebraically.

Simplify $4x(3x^2 - 8x + 5)$.

$$4x (3x^2 - 8x + 5)$$

$$4x(3x^2) - 4x(8x) + 4x(5)$$

$$12x^3 - 32x^2 + 20x$$

Simplify $3x(x - 9) + 2(x - 4)$.

$$3x(x - 9) + 2(x - 4)$$

Distribute.

$$3x(x) - 3x(9) + 2(x) - 2(4)$$

$$3x^2 - 27x + 2x - 8$$

Combine like terms.

$$3x^2 - 25x - 8$$

Multiplying Binomials Using Algebra Tiles

Multiply $(x + 2)$ and $(x + 4)$.

$$(x + 2)(x + 4)$$

$$x^2 + 6x + 8$$

Multiply $(x - 2)$ and $(x + 1)$.

$$(x - 2)(x + 1)$$

$$x^2 - 2x + x - 2$$

$$x^2 - x - 2$$
Multiplying Binomials Using the FOIL Method

Multiplying binomials involves the use of the Distributive Property and the properties of exponents. When multiplying binomials, we need to use the Distributive Property twice. This method of distributing twice is called the FOIL method.

Compare the arrangement for \((x – 3)(x + 5)\) to the FOIL method.

Multiply \((x + 2)\) and \((x + 11)\).

1. Multiply the First terms.
2. Multiply the Outer terms.
3. Multiply the Inner terms.
4. Multiply the Last terms.
5. Combine like terms.

\[ (x + 2)(x + 11) \]

\[ (x + 2)(x + 11) \]

\[ x^2 + 11x + 2x + 22 \]

\[ x^2 + 13x + 22 \]
Special Products

The product of the square of a binomial and the product of a sum and a difference have a specific pattern.

Simplify the square of a binomial.

Answer the following questions to simplify \((2x + 3)^2\).

What is the square of the first term? \((2x)^2 = 4x^2\)

What is twice the product of the two terms? \(2(2x)(3) = 6x\)

What is the square of the second term? \((3)^2 = 9\)

What is the simplified polynomial? \(4x^2 + 12x + 9\)

**Key Point**

The result of the square of a binomial is called a _perfect square trinomial._

Simplify the difference of two squares.

When the two binomials are multiplied the _two middle terms have a sum of 0_ and the _result is the difference of two perfect squares._

\[
(x + 3)(x - 3) = x^2 - 3x + 3x - 9 = x^2 - 9 \\
(8n + 1)(8n - 1) = 64n^2 - 8n + 8n - 1 = 64n^2 - 1
\]
Multiplying Polynomials

Multiply \((2x + 3)\) and \((x^2 - 4x + 1)\).

\[
(2x + 3)(x^2 - 4x + 1)
\]

\[
2x(x^2) - 2x(4x) + 2(x) + 3(x^2) - 3(4x) + 3(1)
\]

\[
2x^3 - 8x^2 + 2x + 3x^2 - 12x + 3
\]

\[
2x^3 - 5x^2 - 10x + 3
\]

Multiply \((x + 1)\), \((x + 2)\), and \((x + 3)\) in two different ways.

Simplify \((x + 1)(x + 2)(x + 3)\).

1. Multiply \((x + 1)\) and \((x + 2)\).

\[
(x + 1)(x + 2) = x^2 + 2x + x + 2
\]

\[
= x^2 + 3x + 2
\]

2. Multiply \((x^2 + 3x + 2)\) and \((x + 3)\).

\[
(x^2 + 3x + 2)(x + 3) = x^3 + 3x^2 + 3x^2 + 9x + 2x + 6
\]

\[
= x^3 + 6x^2 + 11x + 6
\]

\((x + 1)(x + 2)(x + 3) = x^3 + 6x^2 + 11x + 6\).

Simplify \((x + 1)(x + 2)(x + 3)\).

1. Multiply \((x + 2)\) and \((x + 3)\).

\[
(x + 2)(x + 3) = x^2 + 3x + 2x + 6
\]

\[
= x^2 + 5x + 6
\]

2. Multiply \((x + 1)\) and \((x^2 + 5x + 6)\).

\[
(x + 1)(x^2 + 5x + 6) = x^3 + 5x^2 + 6x + x^2 + 5x + 6
\]

\[
= x^3 + 6x^2 + 11x + 6
\]

\((x + 1)(x + 2)(x + 3) = x^3 + 6x^2 + 11x + 6\).
Factoring Polynomials: Lesson Summary with Examples

To factor using the greatest common factor
1. Identify the GCF.
2. Rewrite each term using the GCF.
3. Write the factored form.

To factor by grouping
1. Group the first two terms and the last two terms.
2. Factor out the GCF from each pair.
3. Factor out the common binomial.

Factoring Using the Greatest Common Factor
Find the GCF of $12x^2y^3$, $18xy^2$, and $30x^3y^3$.

$12x^2y^3 = 2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot y \cdot y \cdot y$
$18xy^2 = 2 \cdot 3 \cdot 3 \cdot x \cdot y \cdot y$
$30x^3y^3 = 2 \cdot 3 \cdot 5 \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$

GCF = $2 \cdot 3 \cdot x \cdot y \cdot y$
GCF = $6xy^2$

Factor $12a^3b^4 + 20a^4b^2$.

$12a^3b^4 = 2 \cdot 2 \cdot 3 \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b$
$20a^4b^2 = 2 \cdot 2 \cdot 5 \cdot a \cdot a \cdot a \cdot a \cdot b \cdot b$

GCF = $2 \cdot 2 \cdot a \cdot a \cdot a \cdot b \cdot b$
GCF = $4a^3b^2$

$12a^3b^4 + 20a^4b^2$
$4a^5b^2(3b^2) + 4a^3b^2(5a)$
$4a^3b^2(3b^2 + 5a)$

The GCF can also be a binomial expression.

Factor $x(x + 8) + 9(x + 8)$.
$x(x + 8) − 9(x + 8)$
$(x + 8)(x − 9)$
Factoring by Grouping

Factor $4ab + 5a + 12b + 15$.

1. Group the first two terms and the last two terms.
   \[(4ab + 5a) + (12b + 15)\]

2. Factor out the GCF from each pair.
   \[a(4b) + a(5) + 3(4b) + 3(5)\]
   \[a(4b + 5) + 3(4b + 5)\]

3. Factor out the common binomial.
   \[(4b + 5)(a + 3)\]

When grouping the terms, be careful with the signs.

Factor $4x^2 + 24x - x - 6$.

\[(4x^2 + 24x) + (-x - 6)\]
\[4x(x + 6) - 1(x + 6)\]
\[(x + 6)(4x - 1)\]

Some binomials may be additive inverses. Be sure to use factors that result in common binomials.

\[
\begin{align*}
&18t - 6st - 7s + 21 \\
&(18t - 6st) + (-7s + 21) \\
&6t(3 - s) - 7(s - 3) \\
&6t(3 - s) + 7(-s + 3) \\
&6t(3 - s) + 7(3 - s) \\
&(6t + 7)(3 - s)
\end{align*}
\]
Factoring Trinomials: $x^2 + bx + c$: Lesson Summary with Examples

To factor using algebra tiles
1. Arrange the tiles into a rectangle. Add zero pairs, if necessary.
2. Identify the factors by looking at the lengths of the rectangle.
3. Write the factored form.

To factor a trinomial in the form $x^2 + bx + c$
1. Identify two numbers with a sum of $b$ and a product of $c$.
2. Write the trinomial as the product of two binomials.

Factoring Using Algebra Tiles
To factor using algebra tiles, create a rectangle in order to identify the factors.

Factor $x^2 + 7x + 10$.

$$(x + 2)(x + 5)$$

Factoring trinomials in the form $x^2 + bx + c$ when $c < 0$ will require adding zero pairs. A zero pair consists of one positive $x$-tile and one negative $x$-tile.

Since $x + (-x) = 0$, we can add zero pairs in order to complete a rectangle without changing the value of the original expression.
Factor $x^2 + 6x - 7$.

Arrange the tiles, and then add a zero pair to complete the rectangle.

\[(x + 7)(x - 1)\]

**Factor Diamonds**

Look at the product of these two binomials.

\[(x + 2)(x + 3) = x^2 + 3x + 2x + 6\]

\[= x^2 + (3 + 2)x + (3 \cdot 2)\]

\[= x^2 + 5x + 6\]

Notice that the coefficient of the linear term is the sum of the two numbers, 2 and 3, while the constant term is the product of the two factors.

Use a factor diamond to help us organize the product and sum, and then identify these numbers.

Find two numbers with a product of 40 and a sum of 14.
Factoring Algebraically

To factor a trinomial in the form $x^2 + bx + c$, determine the two numbers with a sum of $b$ and a product of $c$. Remember, you can check your factors by multiplying the two binomials.

\[
\begin{align*}
\text{Factor } x^2 + 2x - 35. \\
&\begin{array}{c}
-35 \\
-5 \\
\_ \\
2
\end{array}
\end{align*}
\]

\[
(x - 5x + 7) \\
x^2 + 7x - 5x - 35 \\
x^2 + 2x - 35 \checkmark
\]

Trinomials with more than one variable that satisfy a specific pattern may be factorable.

Look at the product $(c + 5d)(c - 8d)$.

\[
\begin{align*}
(c + 5d)(c - 8d) &= c^2 - 8cd + 5cd - 40d^2 \\
&= c^2 - 3cd - 40d^2
\end{align*}
\]

The first term includes the square of one of the variables while the last term includes the square of the other variable. The middle term includes the product of both variables. So an expression in this form may be factorable.
Factoring Trinomials: $ax^2 + bx + c$: Lesson Summary with Examples

To factor using algebra tiles

1. Arrange the tiles into a rectangle. Add zero pairs, if necessary.
2. Identify the factors by looking at the lengths of the rectangle.
3. Write the factored form.

To factor a trinomial in the form $ax^2 + bx + c$

1. Identify two numbers with a product of $ac$ and a sum of $b$.
2. Rewrite $bx$ using these two numbers.
3. Factor the expression by grouping.

Factoring Using Algebra Tiles

To factor using algebra tiles, create a rectangle in order to identify the factors.

Factor $4x^2 + 8x + 3$.

\[
\begin{array}{c}
2x + 3 \\
2x + 1 \\
4x^2 + 8x + 3 \\
\end{array}
\]

$(2x + 1)(2x + 3)$
Using Zero Pairs
Recall when using algebra tiles to factor that there may be times when it is necessary to add zero pairs.

Factor $5x^2 + 4x - 1$.

Arrange tiles into a rectangle.

Add zero pairs.

Write the factored form.

Factoring Algebraically
When factoring trinomials in the form $ax^2 + bx + c$ algebraically, we need to also consider the leading coefficient, $a$. To factor, find two numbers whose product is equal to $ac$ and whose sum is equal to $b$, and then factor by grouping.

Factor $12x^2 + 11x + 2$.

Remember, always first look for a GCF when factoring.

Factor $22x^2 - 11x + 66$.
Factoring Special Cases: Lesson Summary with Examples

### Perfect Square Trinomial

\[ a^2 + 2ab + b^2 = (a + b)^2 \]
\[ a^2 - 2ab + b^2 = (a - b)^2 \]

### Difference of Two Squares

\[ a^2 - b^2 = (a - b)(a + b) \]

**Key Point**

When completely factoring polynomials:
- Always factor out a GCF.
- For a binomial, determine if it is the difference of two squares.
- For a trinomial, determine if it is a factorable trinomial or a perfect square trinomial.
- For a polynomial with 4 terms, factor by grouping.

---

**Perfect Square Trinomials**

A perfect square trinomial is a trinomial whose factors are two identical binomials.

Use the following questions to identify whether or not \( x^2 + 18x + 81 \) is a perfect square trinomial.

- **Is the first term a perfect square?** Yes, since \( x^2 = (x)^2 \).
- **Is the last term a perfect square?** Yes, since \( 81 = (9)^2 \).
- **Is the middle term twice the product of the square root of each term?** Yes, since \( 18x = 2(x)(9) \).
- **What is the factored form?** Using \( a^2 + 2ab + b^2 = (a + b)^2 \), \( x^2 + 2(x)(9) + 9^2 = (x + 9)^2 \). So, the factored form is \( (x + 9)^2 \).

Once the trinomial has been identified as a perfect square trinomial, the sign of the middle term will determine the sign of the binomial.

\[ x^2 + 18x + 81 = (x + 9)^2 \]
\[ x^2 - 18x + 81 = (x - 9)^2 \]
Difference of Two Squares

The difference of two squares can be factored as the product of the sum and difference of two binomials.

Use the following statements to factor $4x^2 - 11$.
- The first term is a perfect square, since $4x^2 = (2x)^2$.
- The last term is a perfect square, since $121 = (11)^2$.

Then use $a^2 - b^2 = (a + b)(a - b)$ to write the factored form.
- Factored form: $(2x + 11)(2x - 11)$.

Factoring Completely

A polynomial is completely factored when there are no common factors and the polynomial is written as the product of polynomials, if possible.

Here are two ways to factor $4x^2 - 100$.

<table>
<thead>
<tr>
<th>Factor the difference of two squares first.</th>
<th>Factor out the GCF first.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4x^2 - 100$</td>
<td>$4x^2 - 100$</td>
</tr>
<tr>
<td>$(2x + 10)(2x - 10)$</td>
<td>$4(x^2 - 25)$</td>
</tr>
<tr>
<td>$(2)(x + 5)(2)(x - 5)$</td>
<td>$4(x + 5)(x - 5)$</td>
</tr>
<tr>
<td>$4(x + 5)(x - 5)$</td>
<td></td>
</tr>
</tbody>
</table>
Solving Quadratic Equations by Factoring: Lesson Summary with Examples

**Zero Product Property**

For any real numbers $a$ and $b$, if $ab = 0$, then $a = 0$ or $b = 0$

To solve a quadratic equation by factoring

1. Write the equation in standard form, if necessary.
2. Completely factor the quadratic expression.
3. Use the Zero Product Property to set each factor equal to zero. Then solve each equation.
4. Check the solutions.

**Zero Product Property**

Solve the quadratic equation $(2x - 7)(x + 4) = 0$ using the Zero Product Property.

Set each factor equal to zero.  
\[ 2x - 7 = 0 \quad \text{or} \quad x + 4 = 0 \]

Solve each equation.  
\[ 2x = 7 \]
\[ x = \frac{7}{2} \quad \text{or} \quad x = -4 \]

Check:  
\[ \left(2 \cdot \frac{7}{2} - 4\right) \left(\frac{7}{2} + 4\right) = 0 \quad \text{or} \quad \left(2 \cdot (-4) - 7\right) \left(-4 + 4\right) = 0 \]
\[ 0 \cdot \frac{15}{2} = 0 \checkmark \quad -15 \cdot 0 = 0 \checkmark \]

**Equations Written in Standard Form**

Solve the quadratic equation $3x^2 + 9x = 0$ by factoring.

1. Completely factor the quadratic expression.

   Factor using the GCF:  
   \[ 3x^2 + 9x = 0 \]
   \[ 3x(x + 3) = 0 \]

2. Use the Zero Product Property to set each factor equal to zero. Then solve each equation.

   \[ 3x = 0 \quad \text{or} \quad x + 3 = 0 \]
   \[ x = 0 \quad \text{or} \quad x = -3 \]

3. Check the solutions.

   \[ 3(0)^2 + 9(0) = 0 \quad \text{or} \quad 3(-3)^2 + 9(-3) = 0 \]
   \[ 0 = 0 \checkmark \quad 0 = 0 \checkmark \]
Equations Not Written in Standard Form

Solve the quadratic equation \( x^2 + 8x - 15 = 10x \) by factoring.

1. Write the quadratic equation in standard form.
   \[
   x^2 + 8x - 15 = 10x
   \]
   \[
   x^2 + 8x - 15 - 10x = 10x - 10x
   \]
   \[
   x^2 - 2x - 15 = 0
   \]
   \[
   (x + 3)(x - 5) = 0
   \]

2. Completely factor the quadratic expression.

3. Use the Zero Product Property to set each factor equal to zero. Then solve each equation.
   \[
   x + 3 = 0 \quad \text{or} \quad x - 5 = 0
   \]
   \[
   x = -3 \quad \text{or} \quad x = 5
   \]

4. Check the solutions.
   \[
   (-3)^2 + 8(-3) - 15 = 10(-3)
   \]
   \[
   -9 - 24 - 15 = -30
   \]
   \[
   -30 = -30 \checkmark
   \]
   \[
   (5)^2 + 8(5) - 15 = 10(5)
   \]
   \[
   25 + 40 - 15 = 50
   \]
   \[
   50 = 50 \checkmark
   \]
Solving Quadratic Equations Using Square Roots: Lesson Summary with Examples

To solve a quadratic equation using square roots
1. Isolate the \( x^2 \) term or squared binomial, if necessary.
2. Take the square root of each side.

Remember to always write the solutions in simplest form.

The nature of the roots of a quadratic equation can be described as
- Two real solutions – rational or irrational
- One real rational solution
- No real solutions

The Square Root Property
Find the roots for each quadratic equation using the Square Root Property.

<table>
<thead>
<tr>
<th>( x^2 = 81 )</th>
<th>( x^2 = 128 )</th>
<th>( x^2 = \frac{25}{36} )</th>
<th>( x^2 = 2.56 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = \pm \sqrt{81} )</td>
<td>( x = \pm \sqrt{128} )</td>
<td>( x = \pm \sqrt{\frac{25}{36}} )</td>
<td>( x = \pm \sqrt{2.56} )</td>
</tr>
<tr>
<td>( x = \pm 9 )</td>
<td>( x = \pm \sqrt{64 \cdot 2} )</td>
<td>( x = \pm \frac{5}{6} )</td>
<td>( x = \pm 1.6 )</td>
</tr>
</tbody>
</table>

Check:
- \((9)^2 = 81 \checkmark\)
- \((-9)^2 = 81 \checkmark\)
- \((8\sqrt{2})^2 = 128 \checkmark\)
- \((-8\sqrt{2})^2 = 128 \checkmark\)
- \(\left(\frac{5}{6}\right)^2 = \frac{25}{36} \checkmark\)
- \(\left(-\frac{5}{6}\right)^2 = \frac{25}{36} \checkmark\)
- \((1.6)^2 = 2.56 \checkmark\)
- \((-1.6)^2 = 2.56 \checkmark\)
Solving Quadratic Equations Not in the Form \(x^2 = n\)

Solve the equation \(-6x^2 + 13 = -11\) using square roots.

- Subtract 13 from each side. \(-6x^2 = -24\)
- Divide each side by \(-6\). \(x^2 = 4\)
- Take the square root of each side. \(\pm \sqrt{4} = x\) \(x = \pm 2\)

Solving Quadratic Equations in the Form \((x + p)^2 = n\)

Solve the equation \((x - 7)^2 = 25\) using square roots.

- Take the square root of each side. \(x - 7 = \pm \sqrt{25}\)
  \(x - 7 = \pm 5\)
- Add 7 to each side. \(x = 7 \pm 5\)
  \(x = 7 + 5\) or \(x = 7 - 5\)
  \(x = 12\) or \(x = 2\)

The solutions are \(x = 12\) or \(x = 2\).

Solving Literal Equations Using Square Roots

Solve the Pythagorean Theorem \(a^2 + b^2 = c^2\) for \(b\).

- Isolate \(b^2\) by subtracting \(a^2\) from each side. \(b^2 = c^2 - a^2\)
- Take the square root of each side. \(b = \pm \sqrt{c^2 - a^2}\)
Solving Quadratic Equations by Completing the Square: Lesson Summary with Examples

To solve a quadratic equation in the form $x^2 + bx + c = 0$ by completing the square:

1. Move the constant.
2. Add $\left(\frac{b}{2}\right)^2$ to each side.
3. Factor the perfect square trinomial as the square of a binomial.
4. Solve for $x$.

A perfect square trinomial in the form $x^2 + bx + \left(\frac{b}{2}\right)^2$ can be factored as $\left(x + \frac{b}{2}\right)^2$.

Remember if the leading coefficient of the quadratic equation is not 1, first divide each term by the coefficient, and then continue to complete the square.

Perfect Square Trinomials

Is $x^2 + x + \frac{1}{4}$ a perfect square trinomial?

Ask the following questions:

1. Is the first term a perfect square? Yes, $x^2$ is a perfect square because it can be written as $x \cdot x$.
2. Is the last term a perfect square? Yes, $\frac{1}{4}$ is a perfect square because it can be written as $\frac{1}{2} \cdot \frac{1}{2}$.
3. Is the middle term twice the product of the roots of the first and last terms? Yes, $x = 2\left(x \cdot \frac{1}{2}\right)$
4. Can the trinomial be written as the square of a binomial? Yes, the trinomial can be written as $\left(x + \frac{1}{2}\right)^2$.

Completing the Square

Find the value of $c$ that completes the square for $x^2 + 10x + c$. Write the perfect square trinomial as the square of a binomial.

Find $c$.

$$c = \left(\frac{10}{2}\right)^2$$

Write the perfect square trinomial.

$$x^2 + 10x + \left(\frac{10}{2}\right)^2$$

Write the square of the binomial.

$$(x + 5)^2$$
Solving by Completing the Square

Solve \( x^2 + 6x - 18 = 0 \) by completing the square.

Move the constant. \( x^2 + 6x = 18 \)

Add \( \left( \frac{b}{2} \right)^2 \) to each side. \( x^2 + 6x + 9 = 18 + 9 \)

Factor the perfect square trinomial. \( (x + 3)^2 = 27 \)

Solve for \( x \).

\[
\begin{align*}
  x + 3 &= \pm \sqrt{27} \\
  x + 3 &= \pm 3\sqrt{3} \\
  x &= -3 \pm 3\sqrt{3}
\end{align*}
\]

Solve \( x^2 - 3x - 9 = 0 \) by completing the square.

Move the constant. \( x^2 - 3x = 9 \)

Add \( \left( \frac{b}{2} \right)^2 \) to each side. \( x^2 - 3x + \frac{9}{4} = 9 + \frac{9}{4} \)

\[
\begin{align*}
  x^2 - 3x + \frac{9}{4} &= \frac{45}{4} \\
  \left( x - \frac{3}{2} \right)^2 &= \frac{45}{4}
\end{align*}
\]

Factor the perfect square trinomial. \( \left( x - \frac{3}{2} \right)^2 = \frac{45}{4} \)

Solve for \( x \).

\[
\begin{align*}
  x - \frac{3}{2} &= \pm \sqrt{\frac{45}{4}} \\
  x - \frac{3}{2} &= \pm \frac{3\sqrt{5}}{2} \\
  x &= \frac{3 \pm 3\sqrt{5}}{2}
\end{align*}
\]
Solving When \( a \neq 1 \)

Solve \( 5x^2 + 15x - 10 = 0 \) by completing the square.

Divide each term by \( a \).

\[
\frac{5x^2}{5} + \frac{15x}{5} - \frac{10}{5} = 0
\]

\[
x^2 + 3x - 2 = 0
\]

Move the constant.

\[
x^2 + 3x = 2
\]

Add \( \left( \frac{b}{2} \right)^2 \) to each side.

\[
x^2 + 3x + \frac{9}{4} = 2 + \frac{9}{4}
\]

\[
x^2 + 3x + \frac{9}{4} = \frac{17}{4}
\]

Factor the perfect square trinomial.

\[
\left( x + \frac{3}{2} \right)^2 = \frac{17}{4}
\]

Solve for \( x \).

\[
x + \frac{3}{2} = \pm \sqrt{\frac{17}{4}}
\]

\[
x + \frac{3}{2} = \pm \frac{\sqrt{17}}{2}
\]

\[
x = \frac{-3 \pm \sqrt{17}}{2}
\]
Solving Quadratic Equations Using the Quadratic Formula: Lesson Summary with Examples

To solve a quadratic equation using the quadratic formula:
1. Write the equation in standard form, if necessary.
2. Identify \(a\), \(b\), and \(c\).
3. Substitute the values into the formula and simplify.

Remember, the discriminant can be used to determine the nature of the solutions of a quadratic equation without solving.

<table>
<thead>
<tr>
<th>Discriminant</th>
<th>Nature of the Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b^2 - 4ac &gt; 0)</td>
<td>two real solutions</td>
</tr>
<tr>
<td>(\text{rational (if the discriminant is a perfect square)})</td>
<td></td>
</tr>
<tr>
<td>(\text{irrational (if the discriminant is not a perfect square)})</td>
<td></td>
</tr>
<tr>
<td>(b^2 - 4ac = 0)</td>
<td>one real rational solution</td>
</tr>
<tr>
<td>(b^2 - 4ac &lt; 0)</td>
<td>no real solution</td>
</tr>
</tbody>
</table>

Using the Quadratic Formula
Solve \(7x^2 - 5x - 2 = 0\).

Identify \(a\), \(b\), and \(c\).

Substitute the values into the quadratic formula.

Simplify.

\[a = 7, \quad b = -5, \quad c = -2\]

\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]

\[x = \frac{-( -5) \pm \sqrt{(-5)^2 - 4(7)(-2)}}{2(7)}\]

\[x = \frac{5 \pm \sqrt{25 + 56}}{14}\]

\[x = \frac{5 \pm \sqrt{81}}{14}\]

\[x = \frac{5 \pm 9}{14}\]

\[x = \frac{5 + 9}{14} \quad \text{or} \quad x = \frac{5 - 9}{14}\]

\[x = 1 \quad \text{or} \quad x = -\frac{2}{7}\]
Solve $3x^2 = -6x + 2$.

Write the equation in standard form.

Identify $a$, $b$, and $c$.

Substitute the values into the formula.

Simplify.

The Discriminant

For each quadratic equation, find the solutions using the quadratic formula and note the discriminant.

$x^2 + 8x - 20 = 0$

$x = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(-20)}}{2(1)}$

$x = \frac{-8 \pm \sqrt{144}}{2}$

$x = 2$ or $x = -10$

The discriminant, 144, is positive and is a perfect square.

The equation has two real rational solutions.

$2x^2 + 5x + 1 = 0$

$x = \frac{-5 \pm \sqrt{(5)^2 - 4(2)(1)}}{2(2)}$

$x = \frac{-5 \pm \sqrt{17}}{4}$

The discriminant, 17, is positive and is not a perfect square.

The equation has two real irrational solutions.

$9x^2 + 6x + 1 = 0$

$x = \frac{-6 \pm \sqrt{(6)^2 - 4(9)(1)}}{2(9)}$

$x = \frac{-6 \pm 0}{18}$

$x = -\frac{1}{3}$

The discriminant is 0.

The equation has one real rational solution.

$-3x^2 + x - 2 = 0$

$x = \frac{-3 \pm \sqrt{(1)^2 - 4(-3)(-2)}}{2(1)}$

$x = \frac{-3 \pm \sqrt{-23}}{2}$

no real solution

The discriminant is -23.

The equation has no real solution.
Real-World Applications
A small company selling backpacks can model their quarterly profit, $P$, using the quadratic equation $P = -b^2 + 86b - 800$, where $b$ represents the backpack selling price. At what price can the backpacks be sold to earn a quarterly profit of $985? 

Quadratic model $P = -b^2 + 86b - 800$

Substitute $P = 985$ in the model. $985 = -b^2 + 86b - 800$

Write the equation in standard form. $0 = -b^2 + 86b - 1785$

Solve the equation using the quadratic formula. 

$$x = \frac{-b \pm \sqrt{(86)^2 - 4(-1)(-1785)}}{2(-1)}$$
$$x = \frac{-86 \pm \sqrt{7396 - 7140}}{-2}$$
$$x = \frac{-86 \pm \sqrt{256}}{-2}$$
$$x = \frac{-86 \pm 16}{-2}$$

$x = \frac{-86 + 16}{-2}$ or $x = \frac{-86 - 16}{-2}$

$x = 35$ or $x = 51$

The company can sell backpacks for either $35 or $51 to earn a quarterly profit of $985.
Creating Quadratic Equations: Lesson Summary with Examples

The four methods for solving a quadratic equation algebraically:

- Factoring can be used to solve quadratic equations with factorable quadratic expressions. Equations must be written in standard form. Use if $ab = 0$, then $a = 0$ or $b = 0$.

- Square roots can be used to solve quadratic equations without a linear term. Equations must be written in the form $x^2 = n$. Use $x = \pm \sqrt{n}$.

- Completing the square can be used to solve any quadratic equation. The leading coefficient of the equation must equal to 1. Use $x^2 + bx + \left(\frac{b}{2}\right)^2 = (x + \frac{b}{2})^2$.

- The Quadratic Formula can be used to solve any quadratic equation. Equations must be written in standard form. Use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Remember to always check the reasonableness of your solution.

Modeling Quadratic Equations

Find two consecutive integers whose product is 56.

1st integer: $x$

2nd integer: $x + 1$

$x(x + 1) = 56$

$x^2 + x = 56$

$x^2 + x - 56 = 0$

$(x + 8)(x - 7) = 0$

$x = -8$ or $x = 7$

1st integer: $-8$  
2nd integer: $-7$

1st integer: $7$  
2nd integer: $8$

The integers are $-8$ and $-7$ or $7$ and $8$. 

Modeling the Area of Figures

**Area of a Rectangle** The length of a rectangle is 2 cm more than the width. If the area is 15 cm², find the dimensions of the rectangle.

Let $w$ represent the width. Let $w + 2$ represent the length. Use $A = lw$ to write the equation and solve.

\[
15 = (w + 2)w \\
15 = w^2 + 2w \\
1 + 15 = w^2 + 2w + 1 \\
16 = (w + 1)^2 \\
\pm\sqrt{16} = w + 1 \\
\pm 4 = w + 1 \\
-1 \pm 4 = w \\
w = -5 \text{ or } w = 3
\]

The length is 5 cm and the width is 3 cm.

**Using the Pythagorean Theorem**

A 5-foot ladder is leaning against a wall. The distance between the bottom of the wall and the ladder is 1 foot shorter than the distance the ladder reaches up the wall.
Volume of a Rectangular Box

**Volume** The volume of a rectangular solid is 60 in³. Its width is 3 inches less than its length and its height is 6 inches. Find the length and width of the figure.

Use \( V = lwh \) to write an equation.

\[ 60 = l(l - 3)(6) \]

Solve the equation to find the length.

\[ 60 = l(l - 3)(6) \]
\[ 60 = 6l^2 - 18l \]
\[ 0 = 6l^2 - 18l - 60 \]
\[ 0 = 6(l^2 - 3l - 10) \]
\[ 0 = l^2 - 3l - 10 \]
\[ 0 = (l - 5)(l + 2) \]
\[ l = 5 \quad \text{or} \quad l = -2 \]

(dimensions can't be negative)

Find the width.

\[ 5 - 3 = 2 \]

The length of the solid is 5 inches and the width is 2 inches.
Graphing Quadratic functions Using Standard Form: Lesson Summary with Examples

- A quadratic function is a function written in the form \( y = ax^2 + bx + c \) where \( a \neq 0 \).
- The form \( y = ax^2 + bx + c \) is also known as the standard form of a quadratic function.
- The graph of a quadratic function is a U-shaped curve called a parabola.
- The vertex of a parabola is lowest or highest point on the curve.
- The axis of symmetry of a parabola is a vertical line that passes through the vertex.
- The minimum value is the \( y \)-coordinate of the vertex when the parabola opens up.
- The maximum value is the \( y \)-coordinate of the vertex when the parabola opens down.

<table>
<thead>
<tr>
<th>Key Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>The range of a quadratic function is</td>
</tr>
<tr>
<td>• All real numbers greater than or equal to the minimum value</td>
</tr>
<tr>
<td>• All real numbers less than or equal to the maximum value</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>The Effect of ( a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given the function ( y = ax^2 )</td>
</tr>
<tr>
<td>• If (</td>
</tr>
<tr>
<td>• If (</td>
</tr>
<tr>
<td>• If ( a &gt; 0 ), the graph opens up.</td>
</tr>
<tr>
<td>• If ( a &lt; 0 ), the graph opens down.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Summary of Key Features</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given the function ( y = ax^2 + c )</td>
</tr>
<tr>
<td>• The domain is all real numbers.</td>
</tr>
<tr>
<td>• The range is ( y \geq c ) when ( a &gt; 0 ) or ( y \leq c ) when ( a &lt; 0 ).</td>
</tr>
<tr>
<td>• The vertex and the ( y )-intercept are both ((0, c)).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>The Effect of ( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given the function ( y = ax^2 + c )</td>
</tr>
<tr>
<td>• If ( c &gt; 0 ), the graph is translated up ( c ) units.</td>
</tr>
<tr>
<td>• If ( c &lt; 0 ), the graph is translated down ( c ) units.</td>
</tr>
</tbody>
</table>
To graph a quadratic function written in standard form:

Step 1: Find the axis of symmetry.
Step 2: Find the vertex.
Step 3: Create a table of values.
Step 4: Graph the parabola.

Key Features of a Quadratic Function

Two key features of a quadratic function are its vertex and axis of symmetry.
Understanding the Effect of $a$

Compare the three graphs.

Graph A
\[ y = 2x^2 \]

Graph B
\[ y = -6x^2 \]

Graph C
\[ y = \frac{1}{4}x^2 \]

Graph B is the narrowest and the $|a|$ is greater than 1. Graph C is the widest and $|a|$ is less than 1.

Graphing $y = ax^2$

The value of $a$ stretches the parent function by a factor of $a$ when $|a| \neq 1$ and reflects the parent function across the $x$-axis when $a < 0$.

Graph $y = 3x^2$.

multiply each $y$-coordinate by 3

$(-1, 3)$
$(0, 0)$
$(1, 3)$

Graph $y = -x^2$.

multiply each $y$-coordinate by $-1$

$(-1, -1)$
$(0, 0)$
$(1, -1)$
Understanding the Effect of $c$ and Graphing $y = x^2 + c$

The value of $c$ translates the parent function up and down the $y$-axis.

Graph $y = x^2 + 2$.
1. Graph the parent function.
2. Translate each point of the parent function up 2 units.
3. Draw the curve.

Finding Key Features Using Standard Form

Find the axis of symmetry and the vertex for $f(x) = x^2 + 2x + 3$.

Identify $a$ and $b$.

$$f(x) = x^2 + 2x + 3$$
$$a = 1, \ b = 2$$

Find the axis of symmetry.

Substitute $a = 1$ and $b = 2$ into $x = -\frac{b}{2a}$.

$$x = -\frac{2}{2(1)}$$
$$x = -1$$

The axis of symmetry is $x = -1$.

Find the vertex.

Substitute $x = -1$ into $f(x) = x^2 + 2x + 3$ to find the $y$-coordinate of the vertex.

$$f(-1) = (-1)^2 + 2(-1) + 3$$
$$f(-1) = 1 - 2 + 3$$
$$f(-1) = 2 \ or \ y = 2$$

The vertex is $(-1, 2)$. 
Graphing \( y = ax^2 + bx + c \)

\[
x = -\frac{b}{2a}
\]

\[
x = -\frac{6}{2(-3)}
\]

\[
x = 1
\]

\[
y = -3(1)^2 + 6(1) + 4
\]

\[
y = 7
\]

\[
y = -3(0)^2 + 6(0) + 4
\]

\[
y = 4
\]

\[
y = -3(2)^2 + 6(2) + 4
\]

\[
y = 4
\]
Real-World Application

**Fencing** Joe has 150 feet of fencing to section off a rectangular part of the backyard for his dog using the house as one side. What is the maximum area of the fenced in section of the backyard?

\[
A = lw \\
A = (150 - 2w)w \\
A = 150w - 2w^2 \\
A = -2w^2 + 150w \\
w = \frac{-b}{2a} \\
w = \frac{-150}{2(-2)} \\
w = 37.5 \\
A = -2w^2 + 150w \\
A = -2(37.5)^2 + 150(37.5) \\
A = 2812.5
\]

The maximum area of the fenced in section is **2812.5 square feet.**
Solving Quadratic Equations by Graphing: *Lesson Summary with Examples*

- The $x$-intercepts of a quadratic function show the solutions of a quadratic equation.
- The $x$-coordinate of the $x$-intercept is called a zero of the function.
- The $x$-intercepts of a quadratic function written in the form $y = (x - p)(x - q)$ are $(p, 0)$ and $(q, 0)$.

<table>
<thead>
<tr>
<th>Quadratic Equation $ax^2 + bx + c = 0$</th>
<th>Graph</th>
<th>Quadratic Function $y = ax^2 + bx + c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 + 4x + 3 = 0$</td>
<td><img src="image1" alt="Graph" /></td>
<td>$f(x) = x^2 + 4x + 3$</td>
</tr>
<tr>
<td>solutions: $x = -1$ or $x = -3$</td>
<td></td>
<td>x-intercepts: $(-1, 0)$ and $(-3, 0)$</td>
</tr>
<tr>
<td>roots: $-1$ or $-3$</td>
<td></td>
<td>zeros: $-1$ and $-3$</td>
</tr>
<tr>
<td>$-x^2 + 6x - 9 = 0$</td>
<td><img src="image2" alt="Graph" /></td>
<td>$f(x) = -x^2 + 6x - 9$</td>
</tr>
<tr>
<td>solution: $x = 3$</td>
<td></td>
<td>x-intercept: $(3, 0)$</td>
</tr>
<tr>
<td>root: $3$</td>
<td></td>
<td>zero: $3$</td>
</tr>
<tr>
<td>$x^2 + 4 = 0$</td>
<td><img src="image3" alt="Graph" /></td>
<td>$f(x) = x^2 + 4$</td>
</tr>
<tr>
<td>solutions: no real solutions</td>
<td></td>
<td>x-intercepts: none</td>
</tr>
<tr>
<td>roots: no real roots</td>
<td></td>
<td>zeros: none</td>
</tr>
</tbody>
</table>
To solve a quadratic equation by graphing:
1. Graph the related quadratic function.
2. Identify the x-intercepts

### Solutions, Roots, x-intercepts and Zeros

<table>
<thead>
<tr>
<th>Solutions</th>
<th>Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solve $x^2 + x - 6 = 0$.</td>
<td>Solve $x^2 + x - 6 = 0$.</td>
</tr>
<tr>
<td>$(x + 3)(x - 2) = 0$</td>
<td>$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-6)}}{2(1)}$</td>
</tr>
<tr>
<td>$x = -3$ or $x = 2$</td>
<td>$x = \frac{-1 \pm \sqrt{1 + 24}}{2}$</td>
</tr>
<tr>
<td>The solutions are $x = -3$ and $x = 2$.</td>
<td>$x = -3$ or $x = 2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x-intercepts</th>
<th>Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph $f(x) = x^2 + x - 6$.</td>
<td>Graph $f(x) = x^2 + x - 6$.</td>
</tr>
<tr>
<td>The x-intercepts are $(-3, 0)$</td>
<td>The zeros are $-3$ and $2$.</td>
</tr>
</tbody>
</table>

The x-intercepts are $(-3, 0)$ and $(2, 0)$. The zeros are $-3$ and $2$. 
Approximating Zeros

Find the approximate zeros of the quadratic function \( y = x^2 + 2x - 4 \).

![](image)

The zeros for the quadratic function are approximately \(-3.24\) and \(1.24\).

Real-World Applications

**High Dive** The height of a diver’s jump can be represented by a quadratic function. The standard height in a platform high dive competition is 10 meters. If diver pushes off the platform with a velocity of 8 meters per second the function that models the diver’s height \( y \) after \( x \) seconds is \( y = -4.9x^2 + 8x + 10 \).

![](image)

- The initial height of the diver (10 meters) is the \( y \)-intercept.
- The maximum height of the diver (=13.27 meters) is the \( y \)-coordinate of the vertex.
- The time it takes the diver to reach the maximum height (≈ 0.82 seconds) is the \( x \)-coordinate of the vertex.
- The time when the diver hits the water (≈ 2.46 seconds) is the \( x \)-intercept.
Graphing Quadratic Functions Using Vertex Form: Lesson Summary with Examples

### Vertex Form of a Quadratic Function

Given the function \( y = a(x - h)^2 + k \) where \( a \neq 0 \)

- The axis of symmetry is \( x = h \).
- The vertex is \( (h, k) \).

### Transformations of \( y = x^2 \)

Given the function \( y = a(x - h)^2 + k \)

- Use \( a \) to
  - Stretch \( y = x^2 \) by a factor of \( a \) when \( a \neq 1 \).
  - Reflect \( y = x^2 \) across the \( x \)-axis when \( a < 0 \).

- Use \( h \) to
  - Translate \( y = x^2 \) right \( h \) units when \( h > 0 \).
  - Translate \( y = x^2 \) left \( h \) units when \( h < 0 \).

- Use \( k \) to
  - Translate \( y = x^2 \) up \( k \) units when \( k > 0 \).
  - Translate \( y = x^2 \) down \( k \) units when \( k < 0 \).

### Key Point

The range of \( y = a(x - h)^2 \) is
- \( y \geq 0 \) when \( a > 0 \)
- \( y \leq 0 \) when \( a < 0 \)

The range of \( y = ax^2 + k \) is
- \( y \geq k \) when \( a > 0 \)
- \( y \leq k \) when \( a < 0 \)
To write a function in vertex form:

Step 1: Move the constant

Step 2: Add \( \left( \frac{b}{2} \right)^2 \) to each side.

Step 3: Factor the perfect square trinomial.

Step 4: Write the function in the form \( y = a(x - h)^2 + k \).

**Vertex Form: The Effect of \( h \)**

<table>
<thead>
<tr>
<th>The Effect of ( h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given the function ( y = (x - h)^2 )</td>
</tr>
<tr>
<td>• If ( h &gt; 0 ), the graph of the parent function moves right ( h ) units.</td>
</tr>
<tr>
<td>• If ( h &lt; 0 ), the graph of the parent function moves left ( h ) units.</td>
</tr>
</tbody>
</table>

Determine the axis of symmetry and vertex of each function.

\[
\begin{align*}
  y &= (x + 2)^2 \\
  h &= -2 \\
  \text{Axis of symmetry: } x &= -2 \\
  \text{Vertex: } &(-2, 0)
\end{align*}
\]

\[
\begin{align*}
  y &= (x - 6)^2 \\
  h &= 6 \\
  \text{Axis of symmetry: } x &= 6 \\
  \text{Vertex: } &(6, 0)
\end{align*}
\]
Vertex Form: The Effect of $k$

The Effect of $k$

Given the function $y = x^2 + k$

- If $k > 0$, the graph of the parent function moves up $k$ units.
- If $k < 0$, the graph of the parent function moves down $k$ units.

Determine the axis of symmetry and vertex of each function.

$$y = x^2 + 2$$

Axis of symmetry: $x = 0$

Vertex: $(0, 2)$

$$y = x^2 - 6$$

Axis of symmetry: $x = 0$

Vertex: $(0, -6)$
Vertex Form: Axis of Symmetry and Vertex

Determine the axis of symmetry and vertex of each function.

\[ y = (x - 6)^2 + 4 \]
\[ h = 6, \ k = 4 \]

Axis of symmetry: \( x = 6 \)
Vertex: \((6, 4)\)

\[ y = (x + 3)^2 - 7 \]
\[ h = -3, \ k = -7 \]

Axis of symmetry: \( x = -3 \)
Vertex: \((-3, -7)\)

\[ y = 3(x - 1)^2 - 2 \]
\[ h = 1, \ k = -2 \]

Axis of symmetry: \( x = 1 \)
Vertex: \((1, -2)\)

\[ y = -(x + 5)^2 + 8 \]
\[ h = -5, \ k = 8 \]

Axis of symmetry: \( x = -5 \)
Vertex: \((-5, 8)\)
Graphing Using Vertex Form

Graph \( y = (x - 3)^2 + 2 \) using a table of values.

1. Identify the vertex.
2. Create a table of values.
3. Plot each point and graph the parabola.

\[
\begin{array}{|c|c|}
\hline
x & y \\
\hline
2 & 3 \\
3 & 2 \\
4 & 3 \\
\hline
\end{array}
\]

\[ y = (x - 3)^2 + 2 \]

Graph \( y = \frac{1}{2}(x + 2)^2 + 1 \) using transformations.

1. Graph the parent function.
2. Identify \( a, h, \) and \( k \). \( a = \frac{1}{2}, \ h = -2, \ k = 1 \)
3. Stretch the graph by a factor of \( \frac{1}{2} \).
4. Translate the graph left 2 units and up 1 unit.

Writing in Vertex Form

Write \( y = x^2 + 4x + 6 \) in vertex form.

1. Move the constant.
2. Add \( \left( \frac{b}{2} \right)^2 \) to each side.
3. Factor the perfect square trinomial.
4. Write the function in the form \( y = a(x - h)^2 + k \).
Linear, Exponential, and Quadratic Models: Lesson Summary with Examples

**Key Point**

A data set models
- A linear function, if the y-values have a common difference or constant first difference
- An exponential function, if the y-values have a common ratio
- A quadratic function, if the y-values have a constant second difference

A system of equations is a set of two or more equations with the same variables. A linear-exponential system contains both a linear equation and an exponential equation. A linear-quadratic system contains both a linear equation and a quadratic equation.

To solve a system of equations by graphing:
1. Graph each function on the same coordinate plane.
2. Find the solution(s).

To solve a system of equations using substitution:
1. Isolate a variable in one of the equations, if necessary.
2. Replace the variable in the second equation by substituting the expression from Step 1. Then solve the new equation.
3. Solve for the other variable in either of the original equations.
4. Check.

To solve a system of equations using elimination:
1. If necessary, multiply one or both equations so that the coefficients of one of the variables will be eliminated when adding or subtracting.
2. Add or subtract the equations to eliminate one of the variables.
3. Solve for the remaining variable in the resulting equation.
4. Solve for the other variable in either of the original equations.
5. Check.
Linear, Exponential, and Quadratic Models

Use the common difference, common ratio, and second difference to determine the type of model represented.

Table 1

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>22</td>
</tr>
</tbody>
</table>

Table 1 has a common difference of 5 and represents a linear function.

Table 2

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>3</td>
<td>54</td>
</tr>
<tr>
<td>4</td>
<td>162</td>
</tr>
</tbody>
</table>

Table 2 has a common ratio of 3 and represents an exponential function.

Table 3

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 3 has a constant second difference of 2 and represents a quadratic function.

Linear Growth versus Exponential Growth

Julia has two payment options for babysitting.

- Option A: $50 for the first week and an increase of $25 each week.
- Option B: $50 for the first week and an increase of 25% each week.

<table>
<thead>
<tr>
<th>Week</th>
<th>Salary (Option A)</th>
<th>Week</th>
<th>Salary (Option B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$125</td>
<td>4</td>
<td>$97.66</td>
</tr>
<tr>
<td>5</td>
<td>$150</td>
<td>5</td>
<td>$122.07</td>
</tr>
<tr>
<td>6</td>
<td>$175</td>
<td>6</td>
<td>$152.59</td>
</tr>
<tr>
<td>7</td>
<td>$200</td>
<td>7</td>
<td>$190.73</td>
</tr>
<tr>
<td>8</td>
<td>$225</td>
<td>8</td>
<td>$238.42</td>
</tr>
<tr>
<td>9</td>
<td>$250</td>
<td>9</td>
<td>$298.02</td>
</tr>
<tr>
<td>10</td>
<td>$275</td>
<td>10</td>
<td>$372.53</td>
</tr>
</tbody>
</table>

Julia’s salary for Option B will exceed her salary for Option A between weeks 7 and 8.
Solving Systems of Linear and Exponential Equations by Graphing

A system of linear and exponential equations can have no solution, one solution, or two solutions.

<table>
<thead>
<tr>
<th>Solutions for a System of Linear and Exponential Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>no solution</strong></td>
</tr>
<tr>
<td>( y = -2x - 1 )</td>
</tr>
<tr>
<td>( y = 0.5^x )</td>
</tr>
</tbody>
</table>

Since the two functions do not intersect, the system has no solution.
Since the two functions intersect at (0, 1), the system has one solution.
Since the two functions intersect at approximately (0.46, 1.37) and (3.31, 9.94), the system has two solutions.

Solving Systems of Linear and Quadratic Equations by Graphing

A system of linear and exponential equations can have no solution, one solution, or two solutions.

<table>
<thead>
<tr>
<th>Solutions for a System of Linear and Quadratic Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>no solution</strong></td>
</tr>
<tr>
<td>( y = -x )</td>
</tr>
<tr>
<td>( y = x^2 + 1 )</td>
</tr>
</tbody>
</table>

Since the two functions do not intersect, the system has no solution.
Since the two functions intersect at (0, -2), the system has one solution.
Since the two functions intersect at approximately (-1.11, -0.55) and (3.61, 1.80), the system has two solutions.
Solving Systems of Linear and Quadratic Equations Algebraically

Solve the system of equations using substitution.

\[
\begin{align*}
y &= x + 3 \\
y &= x^2 - x - 12
\end{align*}
\]

\[
\begin{align*}
x + 3 &= x^2 - x - 12 \\
0 &= x^2 - 2x - 15 \\
0 &= (x - 5)(x + 3) \\
x &= 5 \text{ or } x = -3
\end{align*}
\]

\[
\begin{align*}
y &= 5 + 3 \\
y &= 8
\end{align*}
\]

\[
\begin{align*}
y &= -3 + 3 \\
y &= 0
\end{align*}
\]

Solutions: (5, 8) and (−3, 0)

Solve the system of equations using elimination.

\[
\begin{align*}
y &= x + 3 \\
y &= x^2 - x - 12
\end{align*}
\]

\[
\begin{align*}
-\frac{y}{\text{}} &= -x - 3 \\
(+) \frac{y}{\text{}} &= x^2 - x - 12 \\
0 &= x^2 - 2x - 15 \\
0 &= (x - 5)(x + 3) \\
x &= 5 \text{ or } x = -3
\end{align*}
\]

\[
\begin{align*}
y &= (5)^2 - 5 - 12 \\
y &= 8 \\
y &= 0
\end{align*}
\]

\[
\begin{align*}
y &= (-3)^2 - (-3) - 12 \\
y &= -3
\end{align*}
\]

Solutions: (5, 8) and (−3, 0)